# Tight Bounds on the Approximability of Almost-satisfiable Horn SAT

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# Motivation

**The only three non-trivial Boolean CSPs for which satisfiability is polynomial time decidable. [Schaefer'78]**

- LIN-mod-2 – linear equations modulo 2
- 2-SAT
- Horn-SAT – a CNF formula where each clause consists of at most one unnegated literal
  - $x_1$, $\overline{x_2}$
  - $\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}$
  - $x_2 \wedge x_4 \to x_5$ (equivalent to $\overline{x_2} \vee \overline{x_4} \vee x_5$)

# Robust algorithms for almost satisfiable instances I

A small $\epsilon$ fraction of constraints of a satisfiable instance were corrupted by noise. Can we still find a good assignment?

### Finding almost satisfying assignments

Given an instance which is $(1-\epsilon)$-satisfiable, can we efficiently find an assignment satisfying $(1 - f(\epsilon) - o(1))$ constraints, where $f(\epsilon) \to 0$ as $\epsilon \to 0$?

# Robust algorithms for almost satisfiable instances II

No for $(1 - \epsilon)$-satisfiable LIN-mod-2.

Yes for $(1 - \epsilon)$-satisfiable 2-SAT.

Yes for $(1 - \epsilon)$-satisfiable Horn-SAT

# Robust algorithms for almost satisfiable instances II

No for $(1-\epsilon)$-satisfiable LIN-mod-2.

- NP-Hard to find a $(1/2+\epsilon)$-satisfying solution. [Håstad'01]

Yes for $(1-\epsilon)$-satisfiable 2-SAT.

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Yes for $(1-\epsilon)$-satisfiable 2-SAT.

- SDP based algorithm finds a $(1-O(\sqrt{\epsilon}))$-satisfying assignment. [CMM'09]
- Tight under Unique Games Conjecture. [KKMO'07]

Yes for $(1-\epsilon)$-satisfiable Horn-SAT

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Yes for $(1-\epsilon)$-satisfiable Horn-SAT

- LP based algorithm finds a $(1-O(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)}))$-satisfying assignment. [Zwick'98]
- For Horn-3SAT, Zwick's algorithm gives a $(1-\frac{1}{\log(1/\epsilon)})$-satisfying solution, losing a exponentially large factor.
- Is it tight?

# Bounds on approximability of almost satisfiable Horn-SAT

Previously known

| | Horn-3SAT | Horn-2SAT |
|---|---|---|
| Approx. Alg. | $1 - O(\frac{1}{\log(1/\epsilon)})$ [Zwick'98] | $1 - 3\epsilon$ [KSTW'00] |
| NP-Hardness | $1 - \epsilon^c$ for some $c < 1$ [KSTW'00] | $1 - 1.36\epsilon$ from Vertex Cover |
| UG-Hardness | | $1 - (2 - \delta)\epsilon$ from Vertex Cover |

# Bounds on approximability of almost satisfiable Horn-SAT

Our result

| | Horn-3SAT | Horn-2SAT |
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| Approx. Alg. | $1 - O(\frac{1}{\log(1/\epsilon)})$ [Zwick'98] | $1 - 2\epsilon$ |
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Why rely on UGC? Isn't there a subexponential time algorithm [ABS'10] for UGC ?

- Even for $(1 - \epsilon)$-satisfiable 2-SAT, the NP-hardness of finding $(1 - \omega_\epsilon(1)\epsilon)$-satisfying assignment is not known without assuming UGC, – while UGC implies the optimal $(1 - \Omega(\sqrt{\epsilon}))$ hardness.
- People also trying to prove UGC these days... [Khot-Moshkovitz'10]

# Overview

**Part I.**

**Theorem**

*Given a* $(1-\epsilon)$*-satisfiable instance for* Horn-2SAT*, it is possible to find a* $(1-2\epsilon)$*-satisfying assignment efficiently.*

**Part II.**

**Theorem**

*There exists absolute constant* $C>0$*, s.t. for every* $\epsilon>0$*, given a* $(1-\epsilon)$*-satisfiable instance for* Horn-3SAT*, it is UG-hard to find a* $(1-\frac{C}{\log(1/\epsilon)})$*-satisfying assignment.*

# Part I

**Theorem**

*Given a $(1-\epsilon)$-satisfiable instance for* Horn-2SAT*, it is possible to find a $(1-2\epsilon)$-satisfying assignment efficiently.*

Go to Part II...

# Warm up – approximation preserving reduction from Vertex Cover to Horn-2SAT

Given a Vertex Cover instance $G = (V, E)$,

- Each variable $x_i$ in Horn-2SAT corresponds a vertex $v_i \in V$.
- For each $e = (v_i, v_j) \in E$, introduce a clause $\overline{x_i} \vee \overline{x_j}$ of weight $\frac{1}{|E|+1}$.
- For each $v_i \in V$, introduce a clause $x_i$ of weight $\frac{1}{(|E|+1)|V|}$.

Observation,

- Exists optimal solution violating no edge clause.
- For this optimal solution, set of violated vertex clauses $\sim$ set of vertices chosen in optimal Vertex Cover solution.

Therefore, $1 - OPT(\text{Horn2SAT}) = OPT(\text{Vertex Cover})/(|E| + 1)$.

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# Robust algorithm for almost-satisfiable Horn-2SAT

In Min Horn-2SAT Deletion problem, the goal is to find a subset of clauses of minimum total weight whose deletion makes the instance satisfiable.

We prove

**Theorem**

*There is a polynomial-time 2-approximation algorithm for* Min Horn-2SAT Deletion *problem.*

This directly implies

**Theorem**

*Given a* $(1-\epsilon)$*-satisfiable instance for* Horn-2SAT*, it is possible to find a* $(1-2\epsilon)$*-satisfying assignment efficiently.*

# Approximation algorithm for Min Horn-2SAT Deletion

Possible clauses in Horn-2SAT

- "True constraint": $x_i$
- "False constraint": $\overline{x_i}$
- "Disjunction constraint": $\overline{x_i} \vee \overline{x_j}$
- "Implication constraint": $x_i \to x_j$ (equivalent to $\overline{x_i} \vee x_j$)

LP Formulation as follows, we have $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}$.

$$
\begin{array}{llll}
\text{min.} & \multicolumn{3}{l}{\sum_{i \in V} w_i^{(T)}(1 - y_i) + \sum_{i \in V} w_i^{(F)} y_i + \sum_{i<j} w_{ij}^{(D)} z_{ij}^{(D)} + \sum_{i \neq j} w_{ij}^{(I)} z_{ij}^{(I)}} \\
\text{s.t.} & z_{ij}^{(D)} & \geq y_i + y_j - 1 & \forall i < j \\
& z_{ij}^{(I)} & \geq y_i - y_j & \forall i \neq j \\
& z_{ij}^{(D)} & \geq 0 & \forall i < j \\
& z_{ij}^{(I)} & \geq 0 & \forall i \neq j \\
& y_i & \in [0, 1] & \forall i \in V
\end{array}
$$

# Approximation algorithm for Min Horn-2SAT Deletion

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LP Formulation as follows, we have $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}$.

$$\begin{array}{lll}
\text{min.} & \multicolumn{2}{l}{\sum_{i \in V} w_i^{(T)}(1-y_i) + \sum_{i \in V} w_i^{(F)} y_i + \sum_{i<j} w_{ij}^{(D)} z_{ij}^{(D)} + \sum_{i \neq j} w_{ij}^{(I)} z_{ij}^{(I)}} \\
\text{s.t.} & z_{ij}^{(D)} \geq \max\{y_i + y_j - 1, 0\} & \forall i < j \\
& z_{ij}^{(I)} \geq \max\{y_i - y_j, 0\} & \forall i \neq j \\
& y_i \in [0,1] & \forall i \in V
\end{array}$$

# Approximation algorithm for Min Horn-2SAT Deletion

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LP Formulation as follows, we have $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}$.

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\begin{aligned}
\text{min.} \quad & \sum_{i \in V} w_i^{(T)}(1 - y_i) + \sum_{i \in V} w_i^{(F)} y_i \\
& \quad + \sum_{i<j} w_{ij}^{(D)} \max\{y_i + y_j - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i - y_j, 0\} \\
\text{s.t.} \quad & y_i \in [0,1] \quad \forall i \in V
\end{aligned}
$$

# Approximation algorithm for Min Horn-2SAT Deletion

Possible clauses in Horn-2SAT

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LP Formulation as follows, we have $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}$.

$$\begin{aligned}
\text{min.} \quad & \mathrm{Val}(f) = \sum_{i \in V} w_i^{(T)}(1 - y_i) + \sum_{i \in V} w_i^{(F)} y_i \\
& \quad + \sum_{i<j} w_{ij}^{(D)} \max\{y_i + y_j - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i - y_j, 0\} \\
\text{s.t.} \quad & f = \{y_i\} \quad \in [0,1]^V
\end{aligned}$$

# Half-integrality and rounding I

**Lemma**

*Given a solution $f = \{y_i\}$, we can efficiently convert $f$ into $f^* = \{y_i^*\}$ such that each $y_i^* \in \{0, 1, 1/2\}$ is half-integral, and $\mathrm{Val}(f^*) \le \mathrm{Val}(f)$.*

**Corollary**

*We can efficiently find an optimal LP solution and all the variables in the solution are half-integral.*

**Rounding**

Given an optimal LP solution $f = \{y_i\}$ which is half-integral, define $f_{\text{int}} = \{x_i\}$ as follows.
For each $i \in V$, let $x_i = 0$ when $y_i \le 1/2$, and $x_i = 1$ when $y_i = 1$.

# Half-integrality and rounding II

Observation

- $x_i \le y_i$ and $1 - x_i \le 2(1 - y_i)$.
- $\max\{x_i + x_j - 1, 0\} \le \max\{y_i + y_j - 1, 0\}$ (by $x_i \le y_i, x_j \le y_j$).
- $\max\{x_i - x_j, 0\} \le 2\max\{y_i - y_j, 0\}$.
  - When $y_i \le y_j$,
    $x_i \le x_j \Rightarrow \max\{x_i - x_j, 0\} = \max\{y_i - y_j, 0\} = 0$.
  - When $y_i > y_j \Rightarrow y_i - y_j \ge 1/2$,
    $\max\{x_i - x_j, 0\} \le 1 \le 2\max\{y_i - y_j, 0\}$.

# Half-integrality and rounding II

Observation

- $x_i \le y_i$ and $1 - x_i \le 2(1 - y_i)$.
- $\max\{x_i + x_j - 1, 0\} \le \max\{y_i + y_j - 1, 0\}$ (by $x_i \le y_i, x_j \le y_j$).
- $\max\{x_i - x_j, 0\} \le 2\max\{y_i - y_j, 0\}$.

Therefore,

$$\begin{aligned}
\mathsf{Val}(f_{\rm int}) &= \sum_{i \in V} w_i^{(T)}(1 - x_i) + \sum_{i \in V} w_i^{(F)} x_i \\
&\quad + \sum_{i<j} w_{ij}^{(D)} \max\{x_i + x_j - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{x_i - x_j, 0\} \\
&\le \sum_{i \in V} w_i^{(T)} 2(1 - y_i) + \sum_{i \in V} w_i^{(F)} y_i \\
&\quad + \sum_{i<j} w_{ij}^{(D)} \max\{y_i + y_j - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} 2\max\{y_i - y_j, 0\} \\
&\le 2\mathsf{Val}(f) = 2\mathrm{OPT}_{\rm LP} \le 2\mathrm{OPT}.
\end{aligned}$$

# Proof of half-integrality lemma I

Given $f = \{y_i\}$, construct pairs of critical points
$W_f = \{(p, 1-p) : 0 \leq p \leq 1/2, \exists i \in V, s.t.\ p = y_i \vee 1-p = y_i\}$.

**Idea.** Iteratively revise $f$, so that $W_f$ contains less "non-half-integral" pairs after each iteration, while not increasing Val($f$). Done when $W_f$ contains no "non-half-integral" pair.

# Proof of half-integrality lemma II

$W_f = \{(p, 1-p) : 0 \le p \le 1/2, \exists i \in V, s.t.\ p = y_i \vee 1-p = y_i\}$

**In each iteration.** Choose a non-half-integral pair $(p, 1-p) \in W_f$ $(0 < p < 1/2)$. Let

$$S = \{i : y_i = p\}, S' = \{i : y_i = 1-p\}.$$

Let $a$ and $b$ be the two "neighbors" of $p$ in $W_f$. I.e., let

$$a = \max\{q < p : (q, 1-q) \in W_f, 0\},$$
$$b = \min\{q > p : (q, 1-q) \in W_f, 1/2\}.$$

Define

$f^{(t)} = \{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1-t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}$.

**Claim**

$\mathrm{Val}(f^{(t)})$ is linear with $t \in [a, b]$.

Exists $\tau \in \{a, b\}$ such that $\mathrm{Val}(f^{(\tau)}) \le \mathrm{Val}(f^{(p)}) = \mathrm{Val}(f)$. Update $f$ by $f^{(\tau)}$, we have one less non-half-integral pair $(p, 1-p)$ in $W_f$.

# Proof of Claim

$$\mathrm{Val}(f^{(t)}) = \sum_{i \in V} w_i^{(T)}(1 - y_i^{(t)}) + \sum_{i \in V} w_i^{(F)} y_i^{(t)}$$
$$+ \sum_{i<j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\}$$
$$f^{(t)} = \{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1 - t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}$$

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Only need to prove $g_1(t) = \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\}$ and $g_2(t) = \max\{y_i^{(t)} - y_j^{(t)}, 0\}$ are linear with $t \in [a, b]$ for any $i, j$.

# Proof of Claim

$$\mathrm{Val}(f^{(t)}) = \sum_{i\in V} w_i^{(T)}(1 - y_i^{(t)}) + \sum_{i\in V} w_i^{(F)} y_i^{(t)}$$
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- $i, j \in V \setminus (S \cup S')$. $g_1$ and $g_2$ are constant functions.

# Proof of Claim

$$\mathrm{Val}(f^{(t)}) = \sum_{i \in V} w_i^{(T)}(1 - y_i^{(t)}) + \sum_{i \in V} w_i^{(F)} y_i^{(t)}$$
$$+ \sum_{i<j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\}$$
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- $i, j \in V \setminus (S \cup S')$. ✓
- $i \in V \setminus (S \cup S'), j \in S \cup S'$,
  - The only “non-linear point” is $t = 1 - y_i$ for $g_1$ and $t = y_i$ for $g_2$ – they are away from $[a, b]$.

# Proof of Claim

$$\mathrm{Val}(f^{(t)}) = \sum_{i\in V} w_i^{(T)}(1 - y_i^{(t)}) + \sum_{i\in V} w_i^{(F)} y_i^{(t)}$$
$$+ \sum_{i<j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i\neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\}$$
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Only need to prove $g_1(t) = \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\}$ and $g_2(t) = \max\{y_i^{(t)} - y_j^{(t)}, 0\}$ are linear with $t \in [a, b]$ for any $i, j$.

- $i, j \in V \setminus (S \cup S')$. ✓
- $i \in V \setminus (S \cup S'), j \in S \cup S'$, or $i \in S \cup S', j \in V \setminus (S \cup S')$. ✓
- $i \in S, j \in S'$ (or $i \in S', j \in S$).
    - $g_1(t) = y_i^{(t)} + y_j^{(t)} - 1 \equiv 0$ is constant function.
    - When $i \in S, j \in S'$, $y_i^{(t)} \leq y_j^{(t)}$, $g_2(t) \equiv 0$ is constant function.
    - When $i \in S', j \in S$, $y_i^{(t)} \geq y_j^{(t)}$, $g_2(t) = y_i^{(t)} - y_j^{(t)}$ is linear function of $t$.

# Proof of Claim

$$\mathrm{Val}(f^{(t)}) = \sum_{i\in V} w_i^{(T)}(1-y_i^{(t)}) + \sum_{i\in V} w_i^{(F)} y_i^{(t)}$$
$$+\sum_{i<j} w_{ij}^{(D)} \max\{y_i^{(t)}+y_j^{(t)}-1,0\} + \sum_{i\neq j} w_{ij}^{(I)} \max\{y_i^{(t)}-y_j^{(t)},0\}$$
$$f^{(t)} = \{y_i^{(t)}=t\}_{i\in S} \cup \{y_i^{(t)}=1-t\}_{i\in S'} \cup \{y_i^{(t)}=y_i\}_{i\in V\setminus(S\cup S')}$$

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Only need to prove $g_1(t)=\max\{y_i^{(t)}+y_j^{(t)}-1,0\}$ and $g_2(t)=\max\{y_i^{(t)}-y_j^{(t)},0\}$ are linear with $t\in[a,b]$ for any $i,j$.

- $i,j\in V\setminus(S\cup S')$. ✓
- $i\in V\setminus(S\cup S'), j\in S\cup S'$, or $i\in S\cup S', j\in V\setminus(S\cup S')$. ✓
- $i\in S, j\in S'$ (or $i\in S', j\in S$). ✓
- $i,j\in S$ (or $i,j\in S'$).
  - When $i,j\in S$, $y_i^{(t)}+y_j^{(t)}<1$, $g_1(t)\equiv 0$ is constant function.
  - When $i,j\in S'$, $y_i^{(t)}+y_j^{(t)}>1$, $g_1(t)=y_i^{(t)}+y_j^{(t)}-1$ is linear function of $t$.
  - $y_i^t=y_j^t$, thus $g_2(t)\equiv 0$ is constant function.

# Proof of Claim

$$\mathrm{Val}(f^{(t)}) = \sum_{i \in V} w_i^{(T)}(1 - y_i^{(t)}) + \sum_{i \in V} w_i^{(F)} y_i^{(t)} + \sum_{i<j} w_{ij}^{(D)} \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\} + \sum_{i \neq j} w_{ij}^{(I)} \max\{y_i^{(t)} - y_j^{(t)}, 0\}$$

$$f^{(t)} = \{y_i^{(t)} = t\}_{i \in S} \cup \{y_i^{(t)} = 1 - t\}_{i \in S'} \cup \{y_i^{(t)} = y_i\}_{i \in V \setminus (S \cup S')}$$

Only need to prove $g_1(t) = \max\{y_i^{(t)} + y_j^{(t)} - 1, 0\}$ and $g_2(t) = \max\{y_i^{(t)} - y_j^{(t)}, 0\}$ are linear with $t \in [a, b]$ for any $i, j$.

- $i, j \in V \setminus (S \cup S')$. ✓
- $i \in V \setminus (S \cup S'), j \in S \cup S'$, or $i \in S \cup S', j \in V \setminus (S \cup S')$. ✓
- $i \in S, j \in S'$ (or $i \in S', j \in S$). ✓
- $i, j \in S$ (or $i, j \in S'$). ✓

Q.E.D.

# Part II

**Theorem**

*There exists absolute constant $C > 0$, s.t. for every $\epsilon > 0$, given a $(1-\epsilon)$-satisfiable instance for* Horn-3SAT, *it is UG-hard to find a $(1 - \frac{C}{\log(1/\epsilon)})$-satisfying assignment.*

# Proof Method

**Theorem** [Raghavendra'08]

*There is a canonical SDP relaxation* $\mathrm{SDP}(\Lambda)$ *for each CSP* $\Lambda$*. Let* $1 > c > s > 0$*. A* $c$ *vs.* $s$ *integrality gap instance for* $\mathrm{SDP}(\Lambda)$ $\Rightarrow$ *UG-hardness of* $(c - \eta)$ *vs.* $(s + \eta)$ *gap-*$\Lambda$ *problem, for every constant* $\eta > 0$*.*

We prove the UG-hardness by showing

**Theorem**

*There is a* $(1 - 2^{-\Omega(k)})$ *vs.* $(1 - 1/k)$ *gap instance for* $\mathrm{SDP}(\text{Horn-3SAT})$*, for every* $k > 1$*.*

# The canonical SDP for Boolean CSPs I

$\mathcal{C}$: The set of constraints over $X = \{x_1, x_2, \cdots, x_n \in \{0,1\}\}$.

For each $C \in \mathcal{C}$, set up a local distribution $\pi_C$ on all truth-assignments $\{\sigma : X_C \to \{0,1\}\}$.

- Introduce scalar variables $\pi_C(\sigma)$ with non-negativity constraints and $\sum_\sigma \pi_C(\sigma) = 1$.

A lifted LP (in Sherali-Adams system).

$$\begin{aligned} \text{max.} \quad & \mathbf{E}_{C \in \mathcal{C}}[\mathbf{Pr}_{\sigma \in \pi_C}[C(\sigma) = 1]] \\ \text{s.t.} \quad & \mathbf{Pr}_{\sigma \in \pi_C}[\sigma(x_i) = b_1 \wedge \sigma(x_j) = b_2] = X_{(x_i,b_1),(x_j,b_2)} \\ & \qquad \forall C \in \mathcal{C}, x_i, x_j \in C, b_1, b_2 \in \{0,1\} \end{aligned}$$

# An example I

**Instance.** Clause 1 : $x_1 \wedge x_2 \rightarrow x_4$, Clause 2: $x_3 \wedge x_4 \rightarrow x_2$.
**Objective.** Maximize
$\frac{1}{2}(\pi_1(x_1, x_2, x_4) + \pi_1(\overline{x_1}, x_2, x_4) + \pi_1(x_1, \overline{x_2}, x_4) + \pi_1(\overline{x_1}, \overline{x_2}, x_4) + \pi_1(\overline{x_1}, x_2, \overline{x_4}) + \pi_1(x_1, \overline{x_2}, \overline{x_4}) + \pi_1(\overline{x_1}, \overline{x_2}, \overline{x_4})) + \frac{1}{2}(\pi_2(x_3, x_4, x_2) + \pi_2(\overline{x_3}, x_4, x_2) + \pi_2(x_3, \overline{x_4}, x_2) + \pi_2(\overline{x_3}, \overline{x_4}, x_2) + \pi_2(\overline{x_3}, x_4, \overline{x_2}) + \pi_2(x_3, \overline{x_4}, \overline{x_2}) + \pi_2(\overline{x_3}, \overline{x_4}, \overline{x_2}))$
**Constraints.**
$\pi_1(\cdot, \cdot, \cdot)$ and $\pi_2(\cdot, \cdot, \cdot)$ form distributions respectively.
$\pi_1(x_1, x_2, x_4) + \pi_1(\overline{x_1}, x_2, x_4) = \pi_2(x_2, x_3, x_4) + \pi_2(x_2, \overline{x_3}, x_4) = X_{(x_2,1),(x_4,1)}$
$\pi_1(x_1, \overline{x_2}, x_4) + \pi_1(\overline{x_1}, \overline{x_2}, x_4) = \pi_2(\overline{x_2}, x_3, x_4) + \pi_2(\overline{x_2}, \overline{x_3}, x_4) = X_{(x_2,0),(x_4,1)}$
$\pi_1(x_1, x_2, \overline{x_4}) + \pi_1(\overline{x_1}, x_2, \overline{x_4}) = \pi_2(x_2, x_3, \overline{x_4}) + \pi_2(x_2, \overline{x_3}, \overline{x_4}) = X_{(x_2,1),(x_4,0)}$

# An example II

$\pi_1(x_1, \overline{x_2}, \overline{x_4}) + \pi_1(\overline{x_1}, \overline{x_2}, \overline{x_4}) = \pi_2(\overline{x_2}, x_3, \overline{x_4}) + \pi_2(\overline{x_2}, \overline{x_3}, \overline{x_4}) = X_{(x_2,0),(x_4,0)}$

$\pi_1(x_1, x_2, x_4) + \pi_1(\overline{x_1}, x_2, x_4) + \pi_1(x_1, x_2, \overline{x_4}) + \pi_1(\overline{x_1}, x_2, \overline{x_4}) = \pi_2(x_2, x_3, x_4) + \pi_2(x_2, \overline{x_3}, x_4) + \pi_2(x_2, x_3, \overline{x_4}) + \pi_2(x_2, \overline{x_3}, \overline{x_4}) = X_{(x_2,1),(x_2,1)}$

$\pi_1(x_1, \overline{x_2}, x_4) + \pi_1(\overline{x_1}, \overline{x_2}, x_4) + \pi_1(x_1, \overline{x_2}, \overline{x_4}) + \pi_1(\overline{x_1}, \overline{x_2}, \overline{x_4}) = \pi_2(\overline{x_2}, x_3, x_4) + \pi_2(\overline{x_2}, \overline{x_3}, x_4) + \pi_2(\overline{x_2}, x_3, \overline{x_4}) + \pi_2(\overline{x_2}, \overline{x_3}, \overline{x_4}) = X_{(x_2,0),(x_2,0)}$

$\pi_1(x_1, x_2, x_4) + \pi_1(\overline{x_1}, x_2, x_4) + \pi_1(x_1, \overline{x_2}, x_4) + \pi_1(\overline{x_1}, \overline{x_2}, x_4) = \pi_2(x_2, x_3, x_4) + \pi_2(x_2, \overline{x_3}, x_4) + \pi_2(\overline{x_2}, x_3, x_4) + \pi_2(\overline{x_2}, \overline{x_3}, x_4) = X_{(x_4,1),(x_4,1)}$

$\pi_1(x_1, x_2, \overline{x_4}) + \pi_1(\overline{x_1}, x_2, \overline{x_4}) + \pi_1(x_1, \overline{x_2}, \overline{x_4}) + \pi_1(\overline{x_1}, \overline{x_2}, \overline{x_4}) = \pi_2(x_2, x_3, \overline{x_4}) + \pi_2(x_2, \overline{x_3}, \overline{x_4}) + \pi_2(\overline{x_2}, x_3, \overline{x_4}) + \pi_2(\overline{x_2}, \overline{x_3}, \overline{x_4}) = X_{(x_4,0),(x_4,0)}$

# The canonical SDP for Boolean CSPs II

**Add vectors.** Introduce $\mathbf{v}_{(x,0)}$ and $\mathbf{v}_{(x,1)}$ corresponding to the events $x = 0$ and $x = 1$.

**Constraints.**

- $\mathbf{v}_{(x,0)} \cdot \mathbf{v}_{(x,1)} = 0$ – mutually exclusive events
- $\mathbf{v}_{(x,0)} + \mathbf{v}_{(x,1)} = \mathbf{I}$ – probability adds up to 1
- $\mathbf{Pr}_{\sigma \in \pi_C}[\sigma(x_i) = b_1 \wedge \sigma(x_j) = b_2] = \mathbf{v}_{(x_i,b_1)} \cdot \mathbf{v}_{(x_j,b_2)}$

**The canonical SDP.**

$$
\begin{aligned}
\text{max.} \quad & \mathbf{E}_{C \in \mathcal{C}}[\mathbf{Pr}_{\sigma \in \pi_C}[C(\sigma) = 1]] \\
\text{s.t.} \quad & \mathbf{v}_{(x_i,0)} \cdot \mathbf{v}_{(x_i,1)} = 0 \\
& \mathbf{v}_{(x_i,0)} + \mathbf{v}_{(x_i,1)} = \mathbf{I} \qquad \forall i \in [n] \\
& \|\mathbf{I}\|^2 = 1 \qquad \forall i \in [n] \\
& \mathbf{Pr}_{\sigma \in \pi_C}[\sigma(x_i) = b_1 \wedge \sigma(x_j) = b_2] = \mathbf{v}_{(x_i,b_1)} \cdot \mathbf{v}_{(x_j,b_2)} \\
& \qquad \forall C \in \mathcal{C}, x_i, x_j \in C, b_1, b_2 \in \{0,1\}
\end{aligned}
$$

# The canonical SDP for Boolean CSPs III

**Simplification.** Define $\mathbf{v}_{(x,1)} = \mathbf{v}_x$, and $\mathbf{v}_{(x,0)} = \mathbf{I} - \mathbf{v}_x$. The canonical SDP is equivalent to

$$\begin{array}{lrl} \text{max.} & \mathbf{E}_{C\in\mathcal{C}}[\mathbf{Pr}_{\sigma\in\pi_C}[C(\sigma)=1]] & \\ \text{s.t.} & (\mathbf{I}-\mathbf{v}_{x_i})\cdot\mathbf{v}_{x_i} = 0 & \forall i\in[n] \\ & \|\mathbf{I}\|^2 = 1 & \forall i\in[n] \\ & \mathbf{Pr}_{\sigma\in\pi_C}[\sigma(x_i)=1\wedge\sigma(x_j)=1] = \mathbf{v}_{x_i}\cdot\mathbf{v}_{x_j} & \forall C\in\mathcal{C}, x_i, x_j\in C \end{array}$$

# The canonical SDP for Boolean CSPs III

**Simplification.** Define $\mathbf{v}_{(x,1)} = \mathbf{v}_x$, and $\mathbf{v}_{(x,0)} = \mathbf{I} - \mathbf{v}_x$. The canonical SDP is equivalent to

$$
\begin{aligned}
\text{max.} \quad & \mathbf{E}_{C \in \mathcal{C}}[\mathbf{Pr}_{\sigma \in \pi_C}[C(\sigma) = 1]] \\
\text{s.t.} \quad & (\mathbf{I} - \mathbf{v}_{x_i}) \cdot \mathbf{v}_{x_i} = 0 \qquad \forall i \in [n] \\
& \|\mathbf{I}\|^2 = 1 \qquad \forall i \in [n] \\
& \mathbf{Pr}_{\sigma \in \pi_C}[\sigma(x_i) = 1 \wedge \sigma(x_j) = 1] = \mathbf{v}_{x_i} \cdot \mathbf{v}_{x_j} \quad \forall C \in \mathcal{C}, x_i, x_j \in C
\end{aligned}
$$

**Comment**

The SDP is stronger than lifted LP in many cases. For 2-SAT, lifted LP has a huge gap 1 *vs.* 3/4, while SDP gives the optimal gap $(1 - \epsilon)$ *vs.* $(1 - O(\sqrt{\epsilon}))$.

# Gap instance

Consider instace $\mathcal{I}_k^{\mathrm{Horn}}$.

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \to x_1$ , | $x_0 \wedge y_0 \to y_1$ |
| Step 2: | $x_1 \wedge y_1 \to x_2$ , | $x_1 \wedge y_1 \to y_2$ |
| Step 3: | $x_2 \wedge y_2 \to x_3$ , | $x_2 \wedge y_2 \to y_3$ |
| | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \to x_{k+1}$ , | $x_k \wedge y_k \to y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

# Gap instance

Consider instace $\mathcal{I}_k^{\text{Horn}}$.

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \rightarrow x_1$ , | $x_0 \wedge y_0 \rightarrow y_1$ |
| Step 2: | $x_1 \wedge y_1 \rightarrow x_2$ , | $x_1 \wedge y_1 \rightarrow y_2$ |
| Step 3: | $x_2 \wedge y_2 \rightarrow x_3$ , | $x_2 \wedge y_2 \rightarrow y_3$ |
| | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \rightarrow x_{k+1}$ , | $x_k \wedge y_k \rightarrow y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

**Observation**

*$\mathcal{I}_k^{\text{Horn}}$ is not satisfiable. Therefore $\text{OPT}(\mathcal{I}_k^{\text{Horn}}) \leq 1 - \Omega(1/k)$.*

# A good solution for lifted LP

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \to x_1$ , | $x_0 \wedge y_0 \to y_1$ |
| Step 2: | $x_1 \wedge y_1 \to x_2$ , | $x_1 \wedge y_1 \to y_2$ |
| Step 3: | $x_2 \wedge y_2 \to x_3$ , | $x_2 \wedge y_2 \to y_3$ |
| | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \to x_{k+1}$ , | $x_k \wedge y_k \to y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

**Observation**

Clauses from two different steps share at most one variable. No need to worry about pairwise margins.

# A good solution for lifted LP

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \rightarrow x_1$ , | $x_0 \wedge y_0 \rightarrow y_1$ |
| Step 2: | $x_1 \wedge y_1 \rightarrow x_2$ , | $x_1 \wedge y_1 \rightarrow y_2$ |
| Step 3: | $x_2 \wedge y_2 \rightarrow x_3$ , | $x_2 \wedge y_2 \rightarrow y_3$ |
| | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \rightarrow x_{k+1}$ , | $x_k \wedge y_k \rightarrow y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

$$\text{loss} = 2\delta$$

| $x_0(y_0)$ | $\pi_C(\sigma)$ |
|---|---|
| 1 | $1-\delta$ |
| 0 | $\delta$ |

$\Rightarrow$

| $x_0 \wedge y_0 \rightarrow x_1(y_1)$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow \quad 1$ | $1-2\delta$ |
| $0 \wedge 1 \rightarrow \quad 0$ | $\delta$ |
| $1 \wedge 0 \rightarrow \quad 0$ | $\delta$ |

# A good solution for lifted LP

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \to x_1$ , | $x_0 \wedge y_0 \to y_1$ |
| Step 2: | $x_1 \wedge y_1 \to x_2$ , | $x_1 \wedge y_1 \to y_2$ |
| Step 3: | $x_2 \wedge y_2 \to x_3$ , | $x_2 \wedge y_2 \to y_3$ |
| | $\ldots$ | |
| Step $k+1$: | $x_k \wedge y_k \to x_{k+1}$ , | $x_k \wedge y_k \to y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

$\text{loss} = 2\delta$

| $x_1 \wedge y_1 \to x_2 (y_2)$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \to \quad 1$ | $1 - 4\delta$ |
| $0 \wedge 1 \to \quad 0$ | $2\delta$ |
| $1 \wedge 0 \to \quad 0$ | $2\delta$ |

$\Leftarrow$

| $x_0 \wedge y_0 \to x_1 (y_1)$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \to \quad 1$ | $1 - 2\delta$ |
| $0 \wedge 1 \to \quad 0$ | $\delta$ |
| $1 \wedge 0 \to \quad 0$ | $\delta$ |

# A good solution for lifted LP

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \to x_1$ , | $x_0 \wedge y_0 \to y_1$ |
| Step 2: | $x_1 \wedge y_1 \to x_2$ , | $x_1 \wedge y_1 \to y_2$ |
| Step 3: | $x_2 \wedge y_2 \to x_3$ , | $x_2 \wedge y_2 \to y_3$ |
| | $\dots$ | |
| Step $k+1$: | $x_k \wedge y_k \to x_{k+1}$ , | $x_k \wedge y_k \to y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

$$\text{loss} = 2\delta$$

| $x_1 \wedge y_1 \to x_2(y_2)$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \to \quad 1$ | $1 - 4\delta$ |
| $0 \wedge 1 \to \quad 0$ | $2\delta$ |
| $1 \wedge 0 \to \quad 0$ | $2\delta$ |

$\Rightarrow$

| $x_2 \wedge y_2 \to x_3(y_3)$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \to \quad 1$ | $1 - 8\delta$ |
| $0 \wedge 1 \to \quad 0$ | $4\delta$ |
| $1 \wedge 0 \to \quad 0$ | $4\delta$ |

# A good solution for lifted LP

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \rightarrow x_1$ , | $x_0 \wedge y_0 \rightarrow y_1$ |
| Step 2: | $x_1 \wedge y_1 \rightarrow x_2$ , | $x_1 \wedge y_1 \rightarrow y_2$ |
| Step 3: | $x_2 \wedge y_2 \rightarrow x_3$ , | $x_2 \wedge y_2 \rightarrow y_3$ |
| | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \rightarrow x_{k+1}$ , | $x_k \wedge y_k \rightarrow y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

$$\text{loss} = 2\delta$$

| $x_k \wedge y_k \rightarrow x_{k+1}$ <br> $x_k \wedge y_k \rightarrow y_{k+1}$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow \quad 1$ | $1 - 2^{k+1}\delta$ |
| $0 \wedge 1 \rightarrow \quad 0$ | $2^k\delta$ |
| $1 \wedge 0 \rightarrow \quad 0$ | $2^k\delta$ |

$\Leftarrow \cdots$

| $x_2 \wedge y_2 \rightarrow x_3(y_3)$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow \quad 1$ | $1 - 8\delta$ |
| $0 \wedge 1 \rightarrow \quad 0$ | $4\delta$ |
| $1 \wedge 0 \rightarrow \quad 0$ | $4\delta$ |

# A good solution for lifted LP

| | | | |
|---|---|---|---|
| Step 0: | $x_0$ | , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \to x_1$ | , | $x_0 \wedge y_0 \to y_1$ |
| Step 2: | $x_1 \wedge y_1 \to x_2$ | , | $x_1 \wedge y_1 \to y_2$ |
| Step 3: | $x_2 \wedge y_2 \to x_3$ | , | $x_2 \wedge y_2 \to y_3$ |
| | | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \to x_{k+1}$ | , | $x_k \wedge y_k \to y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ | , | $\overline{y_{k+1}}$ |

$$\text{loss} = 2\delta + 2(1 - 2^{k+1}\delta)$$

| $x_k \wedge y_k \to x_{k+1}$ <br> $x_k \wedge y_k \to y_{k+1}$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \to 1$ | $1 - 2^{k+1}\delta$ |
| $0 \wedge 1 \to 0$ | $2^k\delta$ |
| $1 \wedge 0 \to 0$ | $2^k\delta$ |

$\Rightarrow$

| $x_{k+1}(y_{k+1})$ | $\pi_C(\sigma)$ |
|---|---|
| 1 | $1 - 2^{k+1}\delta$ |
| 0 | $2^{k+1}\delta$ |

# A good solution for lifted LP

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \to x_1$ , | $x_0 \wedge y_0 \to y_1$ |
| Step 2: | $x_1 \wedge y_1 \to x_2$ , | $x_1 \wedge y_1 \to y_2$ |
| Step 3: | $x_2 \wedge y_2 \to x_3$ , | $x_2 \wedge y_2 \to y_3$ |
| | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \to x_{k+1}$ , | $x_k \wedge y_k \to y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

$\mathrm{loss} = 2\delta + 2(1 - 2^{k+1}\delta) = 1/2^k$ (by choosing $\delta = 1/2^{k+1}$) .

| $x_k \wedge y_k \to x_{k+1}$ <br> $x_k \wedge y_k \to y_{k+1}$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \to 1$ | $1 - 2^{k+1}\delta$ |
| $0 \wedge 1 \to 0$ | $2^k\delta$ |
| $1 \wedge 0 \to 0$ | $2^k\delta$ |

$\Rightarrow$

| $x_{k+1}(y_{k+1})$ | $\pi_C(\sigma)$ |
|---|---|
| 1 | $1 - 2^{k+1}\delta$ |
| 0 | $2^{k+1}\delta$ |

# Add vectors to get a good SDP solution

**Problem.** Is there any set of vectors corresponding to the following distribution?

| $x_i \wedge y_i \rightarrow x_{i+1} (y_{i+1})$ | | $\pi_C(\sigma)$ |
|---|---|---|
| $1 \wedge 1 \rightarrow$ | $1$ | $1 - 2\delta$ |
| $0 \wedge 1 \rightarrow$ | $0$ | $\delta$ |
| $1 \wedge 0 \rightarrow$ | $0$ | $\delta$ |

# Add vectors to get a good SDP solution

**Problem.** Is there any set of vectors corresponding to the following distribution?

| $x_i \wedge y_i \rightarrow x_{i+1} (y_{i+1})$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow \quad 1$ | $1 - 2\delta$ |
| $0 \wedge 1 \rightarrow \quad 0$ | $\delta$ |
| $1 \wedge 0 \rightarrow \quad 0$ | $\delta$ |

The required inner-product matrix – should be PSD.

| | $\mathbf{I}$ | $\mathbf{v}_{x_i}$ | $\mathbf{v}_{y_i}$ | $\mathbf{v}_{x_{i+1}}$ | $\mathbf{v}_{y_{i+1}}$ |
|---|---|---|---|---|---|
| $\mathbf{I}$ | $1$ | $1-\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{x_i}$ | $1-\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{y_i}$ | $1-\delta$ | $1-2\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{x_{i+1}}$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | |
| $\mathbf{v}_{y_{i+1}}$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | | $1-2\delta$ |

# Add vectors to get a good SDP solution

**Problem.** Is there any set of vectors corresponding to the following distribution?

| $x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$ | | $\pi_C(\sigma)$ |
|---|---|---|
| $1 \wedge 1 \rightarrow$ | $1$ | $1-2\delta$ |
| $0 \wedge 1 \rightarrow$ | $0$ | $\delta$ |
| $1 \wedge 0 \rightarrow$ | $0$ | $\delta$ |

The required inner-product matrix – should be PSD.

| | $\mathbf{I}$ | $\mathbf{v}_{x_i}$ | $\mathbf{v}_{y_i}$ | $\mathbf{v}_{x_{i+1}}$ | $\mathbf{v}_{y_{i+1}}$ |
|---|---|---|---|---|---|
| $\mathbf{I}$ | $1$ | $1-\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{x_i}$ | $1-\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{y_i}$ | $1-\delta$ | $1-2\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{x_{i+1}}$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | ? |
| $\mathbf{v}_{y_{i+1}}$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | ? | $1-2\delta$ |

# Add vectors to get a good SDP solution

**Problem.** Is there any set of vectors corresponding to the following distribution?

| $x_i \wedge y_i \to x_{i+1}(y_{i+1})$ | | $\pi_C(\sigma)$ |
|---|---|---|
| $1 \wedge 1 \to$ | $1$ | $1-2\delta$ |
| $0 \wedge 1 \to$ | $0$ | $\delta$ |
| $1 \wedge 0 \to$ | $0$ | $\delta$ |

The required inner-product matrix – should be PSD.

| | $\mathbf{I}$ | $\mathbf{v}_{x_i}$ | $\mathbf{v}_{y_i}$ | $\mathbf{v}_{x_{i+1}}$ | $\mathbf{v}_{y_{i+1}}$ |
|---|---|---|---|---|---|
| $\mathbf{I}$ | $1$ | $1-\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{x_i}$ | $1-\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{y_i}$ | $1-\delta$ | $1-2\delta$ | $1-\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{x_{i+1}}$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ |
| $\mathbf{v}_{y_{i+1}}$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ | $1-2\delta$ |

# The matrix is PSD because...

$$
\begin{bmatrix}
1 & 1-\delta & 1-\delta & 1-2\delta & 1-2\delta \\
1-\delta & 1-\delta & 1-2\delta & 1-2\delta & 1-2\delta \\
1-\delta & 1-2\delta & 1-\delta & 1-2\delta & 1-2\delta \\
1-2\delta & 1-2\delta & 1-2\delta & 1-2\delta & \color{red}{1-2\delta} \\
1-2\delta & 1-2\delta & 1-2\delta & \color{red}{1-2\delta} & 1-2\delta
\end{bmatrix}
$$

$$
= \quad (1-2\delta)\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix} + \delta \begin{bmatrix}
2 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

**Bad news.** This is the only way to make the matrix PSD.

# Why is that news bad?

From Step $i$

| $x_i \wedge y_i \rightarrow x_{i+1} (y_{i+1})$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow 1$ | $1 - 2\delta$ |
| $0 \wedge 1 \rightarrow 0$ | $\delta$ |
| $1 \wedge 0 \rightarrow 0$ | $\delta$ |

to Step $i + 1$

| $x_{i+1} \wedge y_{i+1} \rightarrow x_{i+2} (y_{i+2})$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow 1$ | ? |
| $0 \wedge 1 \rightarrow 0$ | ? |
| $1 \wedge 0 \rightarrow 0$ | ? |

# Why is that news bad?

From Step $i$

| $x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow$ 1 | $1-2\delta$ |
| $0 \wedge 1 \rightarrow$ 0 | $\delta$ |
| $1 \wedge 0 \rightarrow$ 0 | $\delta$ |

to Step $i+1$

| $x_{i+1} \wedge y_{i+1} \rightarrow x_{i+2}(y_{i+2})$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow$ 1 | $1-2\delta$ |
| $0 \wedge 1 \rightarrow$ 0 | ? |
| $1 \wedge 0 \rightarrow$ 0 | ? |

The inner-product $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}}$ is too large – angle between two vectors is 0.
The probability $\mathbf{Pr}[x_{i+2} = 1]$ ($\mathbf{Pr}[y_{i+2} = 1]$) cannot decrease.

# Being less greedy – decrease the norm slower I

| $x_i \wedge y_i \rightarrow x_{i+1}(y_{i+1})$ | | $\pi_C(\sigma)$ |
|---|---|---|
| $1 \wedge 1 \rightarrow$ | $1$ | $1 - 1.2\delta$ |
| $0 \wedge 1 \rightarrow$ | $0$ | $0.2\delta$ |
| $1 \wedge 0 \rightarrow$ | $0$ | $0.2\delta$ |
| $0 \wedge 0 \rightarrow$ | $1$ | $0.1\delta$ |
| $0 \wedge 0 \rightarrow$ | $0$ | $0.7\delta$ |

The corresponding inner-product matrix.

| | $\mathbf{I}$ | $\mathbf{v}_{x_i}$ | $\mathbf{v}_{y_i}$ | $\mathbf{v}_{x_{i+1}}$ | $\mathbf{v}_{y_{i+1}}$ |
|---|---|---|---|---|---|
| $\mathbf{I}$ | $1$ | $1-\delta$ | $1-\delta$ | $1-1.1\delta$ | $1-1.1\delta$ |
| $\mathbf{v}_{x_i}$ | $1-\delta$ | $1-\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.2\delta$ |
| $\mathbf{v}_{y_i}$ | $1-\delta$ | $1-1.2\delta$ | $1-\delta$ | $1-1.2\delta$ | $1-1.2\delta$ |
| $\mathbf{v}_{x_{i+1}}$ | $1-1.1\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.1\delta$ | $1-1.2\delta$ |
| $\mathbf{v}_{y_{i+1}}$ | $1-1.1\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.1\delta$ |

# The matrix is PSD because...

$$
\begin{bmatrix}
1 & 1-\delta & 1-\delta & 1-1.1\delta & 1-1.1\delta \\
1-\delta & 1-\delta & 1-1.2\delta & 1-1.2\delta & 1-1.2\delta \\
1-\delta & 1-1.2\delta & 1-\delta & 1-1.2\delta & 1-1.2\delta \\
1-1.1\delta & 1-1.2\delta & 1-1.2\delta & 1-1.1\delta & 1-1.2\delta \\
1-1.1\delta & 1-1.2\delta & 1-1.2\delta & 1-1.2\delta & 1-1.1\delta
\end{bmatrix} =
$$

$$
(1-1.2\delta)\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix} + \delta \begin{bmatrix}
1.2 & 0.2 & 0.2 & 0.1 & 0.1 \\
0.2 & 0.2 & 0 & 0 & 0 \\
0.2 & 0 & 0.2 & 0 & 0 \\
0.1 & 0 & & 0.1 & 0 \\
0.1 & 0 & 0 & 0 & 0.1
\end{bmatrix}
$$

# Being less greedy – decrease the norm slower II

The corresponding inner-product matrix.

| | $\mathbf{I}$ | $\mathbf{v}_{x_i}$ | $\mathbf{v}_{y_i}$ | $\mathbf{v}_{x_{i+1}}$ | $\mathbf{v}_{y_{i+1}}$ |
|---|---|---|---|---|---|
| $\mathbf{I}$ | $1$ | $1-\delta$ | $1-\delta$ | $1-1.1\delta$ | $1-1.1\delta$ |
| $\mathbf{v}_{x_i}$ | $1-\delta$ | $1-\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.2\delta$ |
| $\mathbf{v}_{y_i}$ | $1-\delta$ | $1-1.2\delta$ | $1-\delta$ | $1-1.2\delta$ | $1-1.2\delta$ |
| $\mathbf{v}_{x_{i+1}}$ | $1-1.1\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.1\delta$ | $1-1.2\delta$ |
| $\mathbf{v}_{y_{i+1}}$ | $1-1.1\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.2\delta$ | $1-1.1\delta$ |

Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - 1.1\delta = 1 - \gamma$.
Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - 1.2\delta = 1 - 1.09\gamma$.

Would be good if $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - 1.2\gamma$.

# Amplify the angle I

**Start point.**
Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - \gamma$.
Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - (1+\tau)\gamma$.

| $x_{i+1} \wedge y_{i+1} \rightarrow x_{i+2}(y_{i+2})$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \rightarrow 1$ | $1-(1+\tau)\gamma$ |
| $0 \wedge 1 \rightarrow 0$ | $\tau\gamma$ |
| $1 \wedge 0 \rightarrow 0$ | $\tau\gamma$ |
| $0 \wedge 0 \rightarrow 1$ | $\tau\gamma$ |
| $0 \wedge 0 \rightarrow 0$ | $(1-2\tau)\gamma$ |

| | $\mathbf{I}$ | $\mathbf{v}_{x_{i+1}}$ | $\mathbf{v}_{y_{i+1}}$ | $\mathbf{v}_{x_{i+2}}$ | $\mathbf{v}_{y_{i+2}}$ |
|---|---|---|---|---|---|
| $\mathbf{I}$ | $1$ | $1-\gamma$ | $1-\gamma$ | $1-\gamma$ | $1-\gamma$ |
| $\mathbf{v}_{x_{i+1}}$ | $1-\gamma$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ |
| $\mathbf{v}_{y_{i+1}}$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ |
| $\mathbf{v}_{x_{i+2}}$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ | $1-\gamma$ | $1-(1+1.5\tau)\gamma$ |
| $\mathbf{v}_{y_{i+2}}$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+1.5\tau)\gamma$ | $1-\gamma$ |

# The matrix is PSD because...

$$\begin{bmatrix} 1 & 1-\gamma & 1-\gamma & 1-\gamma & 1-\gamma \\ 1-\gamma & 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma \\ 1-\gamma & 1-(1+\tau)\gamma & 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma \\ 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma & 1-\gamma & \color{red}{1-(1+1.5\tau)\gamma} \\ 1-\gamma & 1-(1+\tau)\gamma & 1-(1+\tau)\gamma & \color{red}{1-(1+1.5\tau)\gamma} & 1-\gamma \end{bmatrix} =$$

$$(1-(1+\tau)\gamma)\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 1+\tau & \tau & \tau & \tau & \tau \\ \tau & \tau & 0 & 0 & 0 \\ \tau & 0 & \tau & 0 & 0 \\ \tau & 0 & 0 & \tau & -0.5\tau \\ \tau & 0 & 0 & -0.5\tau & \tau \end{bmatrix}$$

# Where...

$$\begin{bmatrix} 1+\tau & \tau & \tau & \tau & \tau \\ \tau & \tau & 0 & 0 & 0 \\ \tau & 0 & \tau & 0 & 0 \\ \tau & 0 & 0 & \tau & -0.5\tau \\ \tau & 0 & 0 & -0.5\tau & \tau \end{bmatrix} = \begin{bmatrix} 2\tau & \tau & \tau & 0 & 0 \\ \tau & \tau & 0 & 0 & 0 \\ \tau & 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 2\tau & 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0.5\tau & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\tau & 0 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0 & 0.5\tau \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5\tau & -0.5\tau \\ 0 & 0 & 0 & -0.5\tau & 0.5\tau \end{bmatrix} + \begin{bmatrix} 1-5\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is PSD when $0 \leq \tau \leq 0.2$.

# Amplify the angle II

**Start point.**

Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - \gamma$.

Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - (1 + \tau)\gamma$.

| $x_{i+1} \wedge y_{i+1} \to x_{i+2}(y_{i+2})$ | $\pi_C(\sigma)$ |
|---|---|
| $1 \wedge 1 \to 1$ | $1 - (1+\tau)\gamma$ |
| $0 \wedge 1 \to 0$ | $\tau\gamma$ |
| $1 \wedge 0 \to 0$ | $\tau\gamma$ |
| $0 \wedge 0 \to 1$ | $\tau\gamma$ |
| $0 \wedge 0 \to 0$ | $(1 - 2\tau)\gamma$ |

| | $\mathbf{I}$ | $\mathbf{v}_{x_{i+1}}$ | $\mathbf{v}_{y_{i+1}}$ | $\mathbf{v}_{x_{i+2}}$ | $\mathbf{v}_{y_{i+2}}$ |
|---|---|---|---|---|---|
| $\mathbf{I}$ | $1$ | $1-\gamma$ | $1-\gamma$ | $1-\gamma$ | $1-\gamma$ |
| $\mathbf{v}_{x_{i+1}}$ | $1-\gamma$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ |
| $\mathbf{v}_{y_{i+1}}$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ |
| $\mathbf{v}_{x_{i+2}}$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ | $1-\gamma$ | $1-(1+1.5\tau)\gamma$ |
| $\mathbf{v}_{y_{i+2}}$ | $1-\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+\tau)\gamma$ | $1-(1+1.5\tau)\gamma$ | $1-\gamma$ |

**Result.**

Norm: $\|\mathbf{v}_{x_{i+2}}\|^2 = \|\mathbf{v}_{y_{i+2}}\|^2 = 1 - \gamma$.

Inner-product: $\mathbf{v}_{x_{i+2}} \cdot \mathbf{v}_{y_{i+2}} = 1 - (1 + 1.5\tau)\gamma$.

# A two-stage block

**Stage 1.** Reduce the norm.

| | |
|---|---|
| from | Norm: $\|\mathbf{v}_{x_i}\|^2 = \|\mathbf{v}_{y_i}\|^2 = 1 - \delta$<br>Inner-product: $\mathbf{v}_{x_i} \cdot \mathbf{v}_{y_i} = 1 - 1.2\delta$ |
| Step $i$ | |
| to | Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - 1.1\delta = 1 - \gamma$<br>Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - 1.2\delta = 1 - (1+\tau)\gamma$ |

**Stage 2.** Amplify the angle (reduce the inner-product).

| | |
|---|---|
| from | Norm: $\|\mathbf{v}_{x_{i+1}}\|^2 = \|\mathbf{v}_{y_{i+1}}\|^2 = 1 - \gamma$<br>Inner-product: $\mathbf{v}_{x_{i+1}} \cdot \mathbf{v}_{y_{i+1}} = 1 - (1+\tau)\gamma$ |
| Step $i+1$ | |
| to | Norm: $\|\mathbf{v}_{x_{i+2}}\|^2 = \|\mathbf{v}_{y_{i+2}}\|^2 = 1 - \gamma$<br>Inner-product: $\mathbf{v}_{x_{i+2}} \cdot \mathbf{v}_{y_{i+2}} = 1 - (1+1.5\tau)\gamma$ |
| $\cdots \quad \cdots$ | |
| Step $i+(r-1)$ | |
| to | Norm: $\|\mathbf{v}_{x_{i+r}}\|^2 = \|\mathbf{v}_{y_{i+r}}\|^2 = 1 - \gamma$<br>Inner-product: $\mathbf{v}_{x_{i+r}} \cdot \mathbf{v}_{y_{i+r}} = 1 - (1+1.5^{r-1}\tau)\gamma < 1 - 1.2\gamma$ |

# Repeat the blocks

Suppose $k = qr$, let $\delta = 1.1^{-q}/1.2$.

| | |
|---|---|
| Step 0 | Norm: $\|\mathbf{v}_{x_0}\|^2 = \|\mathbf{v}_{y_0}\|^2 = 1 - \delta$<br>Inner-product: $\mathbf{v}_{x_0} \cdot \mathbf{v}_{y_0} = 1 - 1.2\delta$ |
| Block 1 | |
| Step $r$ | Norm: $\|\mathbf{v}_{x_r}\|^2 = \|\mathbf{v}_{y_r}\|^2 = 1 - 1.1\delta$<br>Inner-product: $\mathbf{v}_{x_r} \cdot \mathbf{v}_{y_r} = 1 - 1.2 \cdot 1.1\delta$ |
| Block 2 | |
| Step $2r$ | Norm: $\|\mathbf{v}_{x_{2r}}\|^2 = \|\mathbf{v}_{y_{2r}}\|^2 = 1 - 1.1^2\delta$<br>Inner-product: $\mathbf{v}_{x_{2r}} \cdot \mathbf{v}_{y_{2r}} = 1 - 1.2 \cdot 1.1^2\delta$ |
| $\cdots$ $\cdots$ | |
| Block $q$ | |
| Step $qr$ | Norm: $\|\mathbf{v}_{x_{qr}}\|^2 = \|\mathbf{v}_{y_{qr}}\|^2 = 1 - 1.1^q\delta$<br>Inner-product: $\mathbf{v}_{x_{qr}} \cdot \mathbf{v}_{y_{qr}} = 1 - 1.2 \cdot 1.1^q\delta = 0$ |
| Step $k + 1 = qr + 1$ | Norm: $\|\mathbf{v}_{x_{k+1}}\|^2 = \|\mathbf{v}_{y_{k+1}}\|^2 = 0$ |

# Repeat the blocks

Suppose $k = qr$, let $\delta = 1.1^{-q}/1.2$.

| | |
|---|---|
| Step 0 | Norm: $\|\mathbf{v}_{x_0}\|^2 = \|\mathbf{v}_{y_0}\|^2 = 1 - \delta$<br>Inner-product: $\mathbf{v}_{x_0} \cdot \mathbf{v}_{y_0} = 1 - 1.2\delta$ |
| Block 1 | |
| Step $r$ | Norm: $\|\mathbf{v}_{x_r}\|^2 = \|\mathbf{v}_{y_r}\|^2 = 1 - 1.1\delta$<br>Inner-product: $\mathbf{v}_{x_r} \cdot \mathbf{v}_{y_r} = 1 - 1.2 \cdot 1.1\delta$ |
| Block 2 | |
| Step $2r$ | Norm: $\|\mathbf{v}_{x_{2r}}\|^2 = \|\mathbf{v}_{y_{2r}}\|^2 = 1 - 1.1^2\delta$<br>Inner-product: $\mathbf{v}_{x_{2r}} \cdot \mathbf{v}_{y_{2r}} = 1 - 1.2 \cdot 1.1^2\delta$ |
| $\cdots$ $\cdots$ | |
| Block $q$ | |
| Step $qr$ | Norm: $\|\mathbf{v}_{x_{qr}}\|^2 = \|\mathbf{v}_{y_{qr}}\|^2 = 1 - 1.1^q\delta$<br>Inner-product: $\mathbf{v}_{x_{qr}} \cdot \mathbf{v}_{y_{qr}} = 1 - 1.2 \cdot 1.1^q\delta = 0$ |
| Step $k + 1 = qr + 1$ | Norm: $\|\mathbf{v}_{x_{k+1}}\|^2 = \|\mathbf{v}_{y_{k+1}}\|^2 = 0$ |

loss $= 2\delta = 2^{-\Omega(k)}$ only from Step 0.

# The SDP gap

Gap instace $\mathcal{I}_k^{\text{Horn}}$.

| | | |
|---|---|---|
| Step 0: | $x_0$ , | $y_0$ |
| Step 1: | $x_0 \wedge y_0 \rightarrow x_1$ , | $x_0 \wedge y_0 \rightarrow y_1$ |
| Step 2: | $x_1 \wedge y_1 \rightarrow x_2$ , | $x_1 \wedge y_1 \rightarrow y_2$ |
| Step 3: | $x_2 \wedge y_2 \rightarrow x_3$ , | $x_2 \wedge y_2 \rightarrow y_3$ |
| | $\cdots$ | |
| Step $k+1$: | $x_k \wedge y_k \rightarrow x_{k+1}$ , | $x_k \wedge y_k \rightarrow y_{k+1}$ |
| Step $k+2$: | $\overline{x_{k+1}}$ , | $\overline{y_{k+1}}$ |

**Observation**

$\mathcal{I}_k^{\text{Horn}}$ *is not satisfiable. Therefore* $\text{OPT}(\mathcal{I}_k^{\text{Horn}}) \leq 1 - \Omega(1/k)$.

**Lemma**

$\text{OPT}_{\text{SDP}}(\mathcal{I}_k^{\text{Horn}}) \geq 1 - 2^{-\Omega(k)}$.

# Thank you

**Questions?**

Venkatesan Guruswami, and Yuan Zhou, *Tight Bounds on the Approximability of Almost-satisfiable Horn SAT and Exact Hitting Set*, SODA 2011 (to appear).