Agenda

Analysis of Boolean Functions

1) Fourier Analysis / Poly. over real numbers
2) Spectral graph theory (hypercube graph)
3) Applying Fourier analysis to property testing

Based on Ryan O'Donnell's lecture written by Sarah Allen

18.434

Analysis of Boolean Functions

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Today and Friday David and I will be discussing the class of functions called Boolean Functions.

$$\mathcal{F} = \{ f \mid f : \{0,1\}^n \to \{0,1\} \}$$

Any function that maps a string of n binary digits to a single bit. We tend to think of "0" as a symbol for False and 1 as a symbol for True. Then all of these functions are functions from vectors in $\mathbb{F}_2^n$ to $\{0,1\}$.

As long the alphabet of the domain is two elements we can use two arbitrary elements. It will be convenient to use

$$\mathcal{F} = \{ f \mid f : \{-1,1\}^n \to \{-1,1\} \}$$

or more generally

$$\mathcal{F} = \{ f \mid f : \{-1,1\}^n \to \mathbb{R} \}$$

One often considers one of the following common Boolean functions.

$f:\{-1,1\}^n \to \{-1,1\}$

Majority$_n(x)$: the result is the decision made by a majority of the inputs. Assuming n is odd $\text{sgn}\left(\sum_{i=1}^{n} x_i\right)$

Parity: $\begin{cases} -1 & \text{odd number of } -1 \text{ inputs} \\ 1 & \text{even " " " "} \end{cases}$

Dictatorship$_i$: The result is the i^{th} input

The Fourier expansion of the majority$_3$ function is

$$\boxed{Maj_3(x) = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_1x_2x_3}$$

which can be verified by arithmatic or verifying each assignment.

The Fourier expansion of

$$Parity_3(x) = x_1x_2x_3 \qquad \text{Remember } \{-1,1\}^n \to \{-1,1\}$$

If asked, $Parity_3(x) = x_1 + x_2 + x_3$ for $\{0,1\}^n \to \{0,1\}$

These are just a few examples but

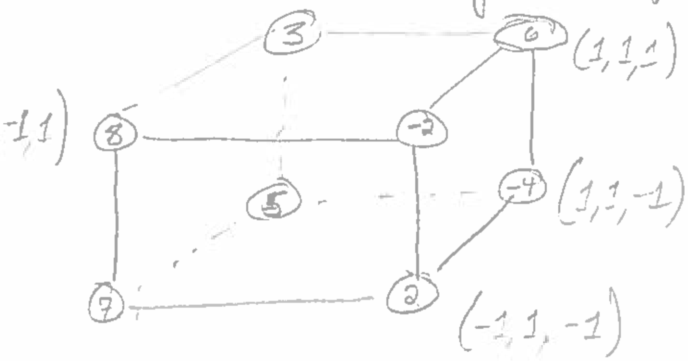
Claim Every function $f:\{-1,1\}^n \to \{-1,1\}$ can be uniquely computed as a multilinear polynomial.

Boolean Functions $\{-1,1\}^n \to \mathbb{R}$

Can be thought of as functions over the vertices of a hypercube. We will consider this running example in three dimensions.

Associating a value $a_v \in \mathbb{R}$ $\forall v \in \{-1,1\}^n$

$f: \{-1,1\}^3 \to \mathbb{R}$



Want to find a poly. that Langrangian interpolates the function, f. For each dimension, i, of the cube we want to consider a term that will be equal to 1 when $x_i = x_i$ is satisfied and 0 when x_j does not satisfied.

Specifically consider constructing a term for the $(1,-1,-1)$ vertex of the 3-dimensional hypercube.

$\left(\frac{1}{2} + \frac{x_1}{2}\right)$ for the first dimension

$\left(\frac{1}{2} - \frac{x_2}{2}\right)$ " " second "

$\left(\frac{1}{2} - \frac{x_3}{2}\right)$ " " third "

~~To ensure the term is no zero only when all three constraints~~ are satisfied, one uses the product. To ensure the function takes on the specified value of f one also has to multiply the product by $f(1,-1,-1)$

Ex.:

$$5 \cdot \left(\frac{1}{2} + \frac{x_1}{2}\right) \cdot \left(\frac{1}{2} - \frac{x_2}{2}\right) \cdot \left(\frac{1}{2} - \frac{x_3}{2}\right) = \text{The } (1,-1,-1) \text{ term of the polynomial}$$

If one sums the terms created for each possible assignment, one creates the polynomial.

One calls this the Fourier expansion.

Pf

Fourier expansion of f

$$f(x) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

Notational Definition

$c_S \triangleq \hat{f}(S)$ This is called the S- Fourier Coefficient of f

There are 2^n subsets, S, $S \subseteq [n]$ which means there must be 2^n monomials. Therefore for every f there are 2^n parameters to select and exactly 2^n coefficients for each monomial. This implies the uniqueness of the fourier expansion.

Notational Definition

$$\chi_S = \prod_{i \in S} x_i \qquad \chi_\phi = 1$$

Note: $\chi_S = \text{Parity}_S(x)$

All Boolean functions are affine combinations of parity functions of the variables.

Spectral Graph Theory

$G \triangleq (V, E)$ Hypercube

n-regular graph

Consider function, f.

$\mathrm{Mean}(f) \triangleq \mathop{\mathbb{E}}_{x \sim \pi}[f(x)]$ where π is a uniform distribution over all vertices $\{-1,1\}^n$.

Definition

f is "Balanced" $\Leftrightarrow$ $\mathrm{Mean}(f) = 0$ Important for Friday's lectures.

$$\mathrm{Var}(f) = \mathop{\mathbb{E}}_{x \sim \pi}\left[f(x)^2\right] - \mathop{\mathbb{E}}_{x \sim \pi}[f(x)]^2$$

This term will always be 1 if $f : \{-1,1\}^n \rightarrow \{-1,1\}$.

Recall that for a vector space and functions f, g the inner product has been defined as

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$$\langle f, g \rangle = \mathop{\mathbb{E}}_{x \sim \pi}[f(x)g(x)]$$

for Boolean Functions $\{-1,1\} \rightarrow \{-1,1\}$

$$= \left(1 - \mathop{P}_{x}(f(x) \neq g(x))\right) - \mathop{P}_{x}(f(x) \neq g(x))$$

if $f(x) = g(x)$ $\langle f, g \rangle = 1$

$$= 1 - 2 \mathop{P}_{x}(f(x) \neq g(x))$$

For use later in property testing

Parity functions form a basis for the vector space

Parity Functions — Eigenfunctions

From S.G.T.

K = normalized adjacency matrix "Markovian Operator"

$L = I - K$ "Laplacian Operator"

Both have the same eigenfunctions and

L's eigenvalues $l_i = 1 - k_i$ where k_i is an eigenvalue of K

$K f(x) = \underset{y \in N(x)}{\mathbb{E}}[f(y)]$ "The expected value of f on y, a random neighbor of x"

Claim 1 Eigenfunctions of K on the hypercube are the parity functions on every subset.

Pf Statement says $\forall S \subseteq [n] \quad \exists k$ s.t. $K \chi_S = k \chi_S$

for a given assignment x and a given vertex x_i ~~of x_i's n neighbors $n-|S|$ have the same value in f,~~ Because the parity$_S$ function does not include them.

$$K \chi_S = \left(\frac{n-2|S|}{n}\right)\chi_S = \left(1 - \frac{2|S|}{n}\right)\chi_S$$

$\left(1 - \frac{2|S|}{n}\right)$ is eigenvalue of χ_S

ranges from -1 to 1

Because this forms a complete basis every function can be a linear combo. of scaled eigenfunctions this is equivalent to what was shown in the Fourier analysis.

Claim 2 The Parity functions for all $S \subseteq [n]$

Pf Consider $S, T \subseteq [n]$

Statement says

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } S \neq T \end{cases}$$

$$\langle \chi_S, \chi_T \rangle = \mathop{\mathbb{E}}_{\pi} \left[\chi_S \chi_T \right]$$ π defined above

$$= \mathop{\mathbb{E}}_{\pi} \left[\prod_{i \in S} x_i \prod_{i \in T} x_i \right]$$

$$= \mathop{\mathbb{E}}_{\pi} \left[\prod_{i \in S \cap T} x_i^2 \prod_{i \in S \Delta T} x_i \right]$$ $S \Delta T$ is the symmetric difference of S and T

$$= \mathop{\mathbb{E}}_{\pi} \left[\prod_{i \in S \cap T} 1 \prod_{i \in S \Delta T} x_i \right]$$ Recall $f : \{-1, 1\}^n \to \{-1, 1\}$

$$= \mathop{\mathbb{E}}_{\pi} \left[\prod_{i \in S \Delta T} x_i \right]$$

Case I $S = T$

$$= \mathop{\mathbb{E}}_{\pi} [1] = 1$$ because $S \Delta T = \phi$

✓

Case II $S \neq T$

$$= \prod_{i \in S \Delta T} \mathop{\mathbb{E}}_{\pi} [x_i]$$ by independence of x_i

$$= \prod_{i \in S \Delta T} 0 = 0$$

✓

□

Now all functions can be uniquely written in this orthogonal basis.

Claim 3 The Fourier Coefficients are exactly the innerproducts of f onto the basis vectors. $\hat{f}(S) = \langle f, \chi_S \rangle \; \forall S \subseteq [n]$

Proof

$$\langle f, \chi_S \rangle = \sum_{T \subseteq [n]} \langle \hat{f}(T)\chi_T, \chi_S \rangle$$

$$= \sum_{T \subseteq [n]} \hat{f}(T) \langle \chi_T, \chi_S \rangle$$

$$= \hat{f}(S) \langle \chi_S, \chi_S \rangle + \sum_{\substack{T \subseteq [n] \\ T \neq S}} \hat{f}(T) \langle \chi_T, \chi_S \rangle$$

$$= \hat{f}(S)$$ By $\langle A, A \rangle = 1$ and the orthogonality of the parity functions

□

The inner product is the correlation between f and parity_S

A Corrolary of Claim 3

Parseval **Thm.** for functions f, g

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$$\langle f, g \rangle = \sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S)$$

Pf

$$\langle f, g \rangle = \mathbb{E}_X \left[f(x) g(x) \right]$$ $f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S$

$$= \mathbb{E}_X \left[\sum_{S,T \subseteq [n]} \hat{f}(S) \chi_S \, \hat{g}(T) \chi_T \right]$$

$$= \mathbb{E}_X \left[\sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S) \right]$$ Orthogonality

$$= \sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S)$$

Important Special Case

Note: $\langle f, f \rangle = 1 = \sum \hat{f}(S)^2$

Property Testing

Querying every possible input to a Boolean function is too costly. One can use a randomized algorithm to test whether a function has a property $\mathcal{P}$. One particularly interesting application is testing a property on a graph and are only able to query whether an edge exists between two vertices. This is made possible when the length of the input to the boolean function, n, is $\binom{|V|}{2}$ of the graph. Notation: If f has property $\mathcal{P}$, $f \in \mathcal{P}$.

When designing randomized algorithms we want the following two properties to be true of the algorithms.

1) If $f \in \mathcal{P}$, algo. outputs Yes with high probability. $(\mathbb{P} > \frac{1}{2})$.

   Algo. is <u>complete</u>.

2) If $f \notin \mathcal{P}$, algo. outputs No with high probability.

   Algo. is <u>sound</u>.

~~For the complexity analysis, we will assume an oracle~~ exists which given an assignment x, will return $f(x)$ in $O(1)$ time

Now there is a complication with the satisfying the soundness condition. Consider the property $\mathcal{P} = \{f \mid f \equiv 1\}$, to verify $f \notin \mathcal{P}$ one would have to query the oracle with all 2^n assignments of x. We introduce a relax the soundness condition in the following way.

<u>Definition</u> f is ϵ-far from $\mathcal{P}$ if $\forall g \in \mathcal{P}$, $\mathbb{P}_x[f(x) = g(x)] > \varepsilon$ or $\langle f, g \rangle < 1 - 2\varepsilon$.

2) If f is ε-far from $\mathcal{P}$, output No with high probability.

   Algo. is sound.

Theorem BLR 90

Property $\mathcal{P}$ where $f \in \mathcal{P}$ iff f is linear function. is testable in $O(\frac{1}{\varepsilon})$ queries

this means $f(x \cdot y) = f(x) f(y)$ where $(x \cdot y)_i = x_i \cdot y_i$. We will propose an algorithm and check that it satisfies both constraints.

Proposed Algorithm: Subroutine 1

1. Select $x, y \sim \{-1, 1\}^n$ independently and uniformly
2. Compute $z = x \cdot y$
3. Query $f(x), f(y), f(z)$
4. If $f(z) \neq f(x) f(y)$, return No

Run subroutine 1 $O(\frac{1}{\varepsilon})$ times.
If none return No, return Yes

If f is linear all of the subroutines will suceed so $\mathbb{P}[\text{Yes}] = 1$.
Algo. is complete.

If f is not linear, the $\mathbb{P}[\text{Yes}] = \mathop{\mathbb{E}}_{x,y}\left[\frac{1}{2} + \frac{1}{2} f(x) f(y) f(z)\right]$

$$\mathbb{P}[\text{Yes}] = \frac{1}{2} + \frac{1}{2} \mathop{\mathbb{E}}_{x,y}\left[\left(\sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i\right)\left(\sum_{T \subseteq [n]} \hat{f}(T) \prod_{i \in T} y_i\right)\left(\sum_{U \subseteq [n]} \hat{f}(U) \prod_{i \in U} z_i\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{S,T,U \subseteq [n]} \hat{f}(S) \hat{f}(T) \hat{f}(U) \mathop{\mathbb{E}}_{x,y}\left[\prod_{i \in S} x_i \prod_{i \in T} y_i \prod_{i \in U} x_i y_i\right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{S,T,U \subseteq [n]} \hat{f}(S) \hat{f}(T) \hat{f}(U) \mathop{\mathbb{E}}_{x,y}\left[\prod_{i \in S \Delta U} x_i \prod_{i \in T \Delta U} y_i\right]$$

Using the symmetric difference trick used in Claim 2

$$= \frac{1}{2} + \frac{1}{2} \sum_{S,T,U \subseteq [n]} \hat{f}(S) \hat{f}(T) \hat{f}(U) \mathop{\mathbb{E}}_{x}\left[\prod_{i \in S \Delta U} x_i\right] \mathop{\mathbb{E}}_{y}\left[\prod_{i \in T \Delta U} y_i\right]$$

Due to the independent selection of x and y.

$\mathbb{P}[\text{Yes}] = \frac{1}{2} + \frac{1}{2} \sum_{S \subseteq [n]} (\hat{f}(S))^3$ Unless $S \triangle U$ and $T \triangle U$ are $\emptyset$ the terms zero.

$\leq \frac{1}{2} + \frac{1}{2} \max_S (\hat{f}(S)) \sum_{S \subseteq [n]} (\hat{f}(S))^2$

$= \frac{1}{2} + \frac{1}{2} \max_S (\hat{f}(S))$ By Parseval's Theorem

$\Rightarrow P[\text{No}] = 1 - \mathbb{P}[\text{Yes}] \geq \frac{1}{2} + \frac{1}{2} \max_S (\hat{f}(S))$ which further implies the algorithm is sound.

The number of queries the algo. makes is $3O\left(\frac{1}{\varepsilon}\right) \in O\left(\frac{1}{\varepsilon}\right)$ $\square$