Expander Graphs

Today we cover definitions and intuition.
Monday we cover application.

- What are expander graphs?
- Define them:
  - edge expanders
  - vertex expanders
  - Spectral expanders

Example: $n=5$ bigraph
Example: $n=$huge Complete graph
<u>sparse</u>
Finding Them: Random Bipartite Expanders
Spectral Expanders and Random Walks
- Other notation for Edge and Vertex

What are expander graphs?

"well connected graphs" "Sparse"

Undirected graph $G(V,E)$, $|V|=n$, we consider huge graphs $n \to \infty$.

The graph is d-regular. $deg(u)=d$ for all $u \in V$

Expander graph has following property:

Edge

For any $S \subseteq V$, $0 < |S| \leq \frac{n}{2}$,

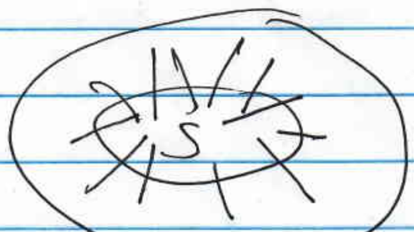
$|E(S,\bar{S})| = \Omega(|S|)$

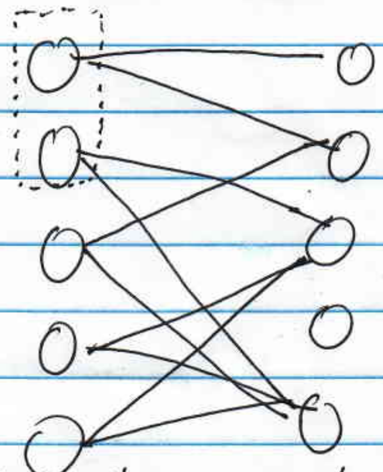
$\Pr_{u \sim v}[u \in S, v \notin S] \geq \varepsilon$ for $\varepsilon > 0$

Take some set of less than half the vertices.

Pick a vertex in S and one not in S.

Odds are there is an edge between them.





Consider left d-regular bipartite graph.

d-regularity in left set. Choose expansion $S \subseteq L$

$d=2$, $n=5$. Look at top 2 nodes as S, 4 edges leave.

Intro to construction.

Usually we want <u>Sparse</u> expanders (more on Monday).

Imagine a complete graph. n vertices, $\frac{n(n-1)}{2}$ edges.
Easy to see $|E(S, \bar{S})| = \Omega(|S|)$ for $0 < |S| < \frac{n}{2}$

But this graph is the opposite of sparse!

~~For applications~~, Random graphs, however, also tend to be good expanders.

To begin, let's start with random bipartite graphs.

Vertex: $\boxed{|N(S)| \geq \varepsilon \cdot d \cdot |S| \text{ for } d\text{-regular graph.}}$

Def A bipartite multigraph G is a (K, A) vertex expander if for all sets S of left-vertices of size at most K the neighborhood $N(S)$ is of size at least $A \cdot |S|$

So now let's see why random Bip are good expanders

Random bipartite expanders

Remember: Defined s.t. $S \subseteq L$
To construct: choose d neighbors from R for each vertex in L.

Parallel edges ok!

Theorem:
Let $Bip_{N,D}$ be the set of bipartite multigraphs that have N vertices on each side. The left side is D-regular -- each node has D neighbors.

For every constant D there exists a constant $\alpha > 0$ s.t. for all N, a uniformly random graph from $Bip_{N,D}$ is an $(\alpha N, D-2)$ vertex expander w/ probability at least $\frac{1}{2}$.

Proof:
For $K \leq \alpha N$ let p_K be Pr there exists a left-set S of size exactly K that does not expand at least $D-2$. Fix S of size K, $N(S)$ is a set of KD random vertices in $[N]$. $v_1, v_2, \ldots, v_{KD}$ were chosen in sequence. Pr v_i ~~is~~ is a repeat $(v_i \in \{v_1 \ldots v_{i-1}\})$ is at most

$$\frac{(i-1)}{N} \leq \frac{KD}{N}$$

Continued

Random Bipartite continued.

$$\Pr[|N(S)| \leq \underbrace{(D-2)\cdot K}] \leq \Pr[\text{there are at least } 2K \text{ repititions}]$$

$(D-2)K$ remember this is a $(D-2)$ expander

ways to choose 2K "repeats"

$$\leq \underbrace{\binom{KD}{2K}} \underbrace{\left(\frac{KD}{N}\right)^{2K}}$$

Pr all these positions become "repeat"

So

$$p_K \leq \binom{N}{K}\binom{KD}{2K}\left(\frac{KD}{N}\right)^{2K}$$

$$\leq \left(\frac{Ne}{K}\right)^K \left(\frac{KDe}{2K}\right)^{2K} \left(\frac{KD}{N}\right)^{2K} = \left(\frac{e^3 D^4 K}{4N}\right)^K$$

Since $K \leq \alpha N$ Set $\alpha = \left(\frac{1}{e^3 D^4}\right)$ to get $p_k \leq 4^{-K}$

So $\Pr_{G \in Bip_{N,D}}[G \text{ is not an } (\alpha N, D-2) \text{ expander}] \leq \sum_{K=1}^{\lceil \alpha N \rceil} 4^{-K} \leq \frac{1}{2}$

So Random Bipartite Multigraphs are likely expanders!

This is actually also true for d-regular graphs

Theorem: For any natural number $d \geq 3$ and large n, there exist d-regular graphs on n vertices which are $\left(\frac{d}{10}, \frac{n}{2}\right)$ edge expanders.

Proof is similar to above so we skip.

thanks wikipedia

Expansion Types: Other notation

Edge expansion: $h(G)$ on G w/ n vertices

$$h(G) = \min_{0<|S|\le \frac{n}{2}} \frac{|\partial S|}{|S|}$$

Where $\partial S := \{(u,v) \in E(G) : u \in S, v \in V(G) \setminus S\}$
∂S is the edge boundary of S. Set of edges w/ exactly one point in S.

Vertex Expansion: on non regular graphs, vertex expansion has 2 parts, $h_{(out)}(G)$ and $h_{(in)}(G)$

$$h_{out}(G) = \min_{0<|S|\le \frac{n}{2}} \frac{|\partial_{out}(S)|}{|S|}$$

$\partial_{out}(S)$ is the outer boundary of S. The set of vertices in $V(G) \setminus S$ w/ at least one neighbor in S.

$$h_{in}(G) = \min_{0<|S|\le \frac{n}{2}} \frac{|\partial_{in}(S)|}{|S|}$$

$\partial_{in}(S)$ is the inner boundary of S. The set of vertices in S with at least one neighbor in $V(G) \setminus S$.

Spectral Expansion

hint: incoming Cheeger's

A graph is well connected means random walks converge to the stationary distribution.

G is a regular digraph w/ spectral expansion $\gamma(G) = 1-\lambda(G)$ for $\gamma \in [0,1]$ when $\lambda(G)$ is the ~~largest~~ second largest eigenvalue of a random walk on G.

$\lambda(G)$ is 2nd largest of transition matrix

So $\gamma(G)$ is second smallest in Laplacian (G)

Def: For $\gamma \in [0,1]$ a regular digraph G has spectral expansion γ if $\gamma(G) \geq \gamma$, $(\lambda(G) \leq 1-\gamma)$

Larger $\gamma(G)$ lead to better expansion.

~~Proof:~~ Return of the Cheeger

- Let G be graph with $V(G)$, $E(G)$. $A \subseteq V(G)$ is a collection of V, let ∂A denote all edges going from A to a vertex outside A.
- The Cheeger Constant $h(G)$ is

$$h(G) = \min\left\{ \frac{|\partial A|}{|A|} : A \subseteq V(G),\ 0 < |A| \leq \tfrac{1}{2}|V(G)| \right\}$$

relating $h(G)$ to $\gamma(G)$:

$$\frac{1}{2}(d-\gamma(G)) \leq h(G) \leq \sqrt{2d(d-\gamma(G))}$$