CSCI-B609: A Theorist's Toolkit, Fall 2016 09/08/2016

Lecture 06: Proof of Cheeger's Inequality (cont.)

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1 Recap

Last time:

- We defined *conductance* as :

$$\Phi(S) = \frac{\text{edges}(S, \bar{S})}{\min\{vol(S), vol(\bar{S})\}}.$$

  For a d-regular graph: $\Phi(S) = \dfrac{\text{edges}(S, \bar{S})}{d \cdot \min(|S|, |\bar{S}|)}$.

  We are interested in finding *Sparsest Cut*

$$\Phi_G = \min_{S:\emptyset \neq S \subsetneq V} \Phi(S) = \min_{S:0<|S|\leq|V|/2} \frac{\text{edges}(S, \bar{S})}{d \cdot |S|}.$$

- Cheeger's inequality states that:

$$\frac{1}{2}\lambda_2 \leq \Phi_G \leq \sqrt{2\lambda_2},$$

  where λ_2 is the second eigenvalue of the normalized Laplacian of G. We proved the lower bound in the previous lecture.

This lecture will be devoted to the proof of the upper bound of Φ_G. We will prove something weaker – $\Phi_G \leq 2\sqrt{\lambda_2}$. The extra $\sqrt{2}$ factor can be saved via a slightly different argument in Section 2.1.

2 Proof of the hard inequality: $\Phi_G \leq 2\sqrt{\lambda_2}$

Idea: In proving the rightmost inequality, we have the $2\sqrt{\lambda_2}$ upperbound and we would like to construct the set S. We observe that the value of λ_2 can be crazy, we would need to make it more regular. We will start with that.

Let us assume $\vec{X} \in \mathbb{R}^n$ certifies λ_2 is small. Let $\vec{X} \perp \vec{1}$ i.e. $\sum_u \vec{X}(u) = 0$. This means it's orthogonal to first vector.

From previous lectures, by the definition of Rayleigh quotient, we know:

$$\frac{\vec{X}^T L_G \vec{X}}{\vec{X}^T \vec{X}} = \lambda_2.$$

We are going to change the assumption about $\vec{X}$ a little bit more. Assume that $\left\|\vec{X}\right\|_2^2 = \vec{X}^T\vec{X} = 1$. so $\vec{X}^T L_G \vec{X} = \lambda_2$. In doing so, We just scaled the vector X. But the eigenvalues are still the same. We would like to use this $\vec{X}$ to construct the set S.

In summary, our assumption about $\vec{X}$: $\vec{X} \perp \vec{1}$ and $\left\|\vec{X}\right\|_2^2 = \vec{X}^T\vec{X} = 1$.

Now, let us look at $\lambda_2 = \vec{X}^T L_G \vec{X} = \frac{1}{d}\sum_{u \sim v}[\vec{X}(u) - \vec{X}(v)]^2$. Intuitively we would want to make $\vec{X}(u) - \vec{X}(v)$ small i.e. the labels u, v to be close. If u, v is scattered, $\vec{X}(u) - \vec{X}(v)$ will be big and λ_2 will be big. Then the question arise: Why not make everything same? But that would be the first Eigenvalue. The constraints we assumed earlier says that somehow the values should be spread out (by assumption, we have sum $\sum \vec{X}(u) = 0$ and **variance** is 1). On the other hand, if we have a relatively denser subgraph, the variance would be small.

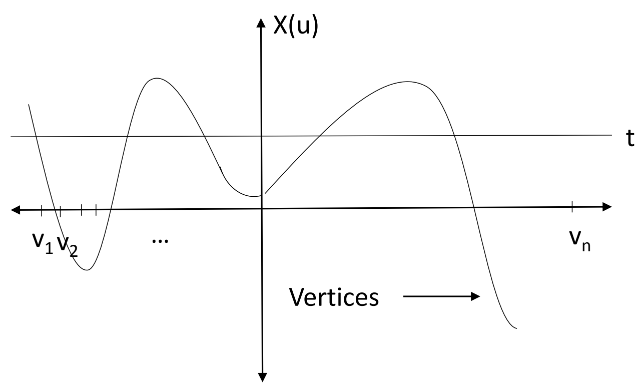

Figure 1: Selecting vertices based on threshold t

Idea: Figure out which vertices are closer to each other (i.e. almost has the same label). These set of vertices will be the members of the set S we are constructing.

Let us denote all the vertices by $v_1, v_2, \cdots, v_n$ and we plot their values $\vec{X}(u)$ (Figure 1). Then we set up a threshold t and we choose all the vertices with $X(u) > t$. This is the simplest way to find out closer vertices.

Definition 1. *Define level set* $S_t = \{u \in x(\vec{u}) > t\}$, *parameterized by threshold* t.

Vertices above threshold t are included in this set.

Question. t can be arbitrary but how many different levels can be possible?

Answer. There are at most $n+1$ identical level sets, including the empty set.

Claim 1. *$\exists\, t$ s.t. $S = S_t$ satisfies* $\dfrac{\text{edges}(S,\bar{S})}{d.|S|} \le \sqrt{2\lambda_2}$

Based on the claim, our algorithm for finding set S goes in the following way:

Algorithm to find S. Try all the level sets. That means based on current t we selected, construct the set S, then compute $\Phi(S)$ and then choose the set S with smallest conductance.

The problem with directly proving Claim 1 is that the constructed set S can be too big. Therefore we need to deal with $d \cdot \min\{|S|, |\bar{S}|\}$ in the denominator of $\Phi(S)$. If we could make sure that $|S| \le |V|/2$, then the denominator becomes $d|S|$ which behaves much more nicely. Therefore we slightly revise our proof plan, and break it into the following two steps.

Definition 2. *Call $\vec{Y} \in \mathbb{R}^v$* convenient *if $\vec{Y}(u) \ge \vec{0}$ and $\#\{u : \vec{Y}(u) > 0\} \le \frac{n}{2}$. Let us call $\{u : \vec{Y}(u) > 0\}$ the* support *of $\vec{Y}$. Therefore the support of $\vec{Y}$ is at most $\frac{n}{2}$.*

Proof Plan.

- Step 1: Find a convenient Y s.t. Rayleigh quotient $\dfrac{\vec{Y}^T L_G \vec{Y}}{\vec{Y}^T \vec{Y}} \le 2\lambda_2$
- Step 2: Show that $\exists\, t \ge 0$ s.t. for $S = S_t(\vec{Y})$, $\dfrac{\text{edges}(S,\bar{S})}{d|S|} \le 2\sqrt{\lambda_2}$.

If we follow step 1 and 2, then S can't be big because by the definition of convenient $\vec{Y}$, atmost $\frac{n}{2}$ entries are positive. And the level set contains only those vertices with values $> t$. So we can have the following comment on the cardinality of set S:

Remark 1. $\#\{u : \vec{Y}(u) > 0\} \le \frac{n}{2}$ *ensures* $|S_t(\vec{Y})| \le \frac{n}{2},\ \forall t \ge 0$.

By guaranteeing a lot of zeroes in vertices, we are trying to ensure that small amount of vertices are beyond the threshold t.

2.1 Step 1

Now let's try to execute step 1 i.e. How to find $\vec{Y}$?

Claim 2. *Given $\vec{X} \perp \vec{1}$ and* $\left\|\vec{X}\right\|_2^2 = \vec{X}^T\vec{X} = 1$, $\exists \vec{Y}, \vec{Z} \in \mathbb{R}^n$

1. *Both* $\vec{Y}, \vec{Z} \geq \vec{0}$*;*
2. $\forall u \in V,\ \vec{Y}(u) \cdot \vec{Z}(u) = 0$*, i.e. the support of* Y *and* Z *do not intersect;*
3. $R(L_G, \vec{Y}), R(L_G, \vec{Z}) \leq 2\lambda_2$.

Proof. For parameter $t \in \mathbb{R}$ (different from previous threshold notation t), set $\vec{Y_t}(u) = \max\{\vec{X}(u) - t, 0\}$ (Figure 2) and $\vec{Z_t}(u) = \max\{t - \vec{X}(u), 0\}$ s.t. items 1 and 2 are satisfied.

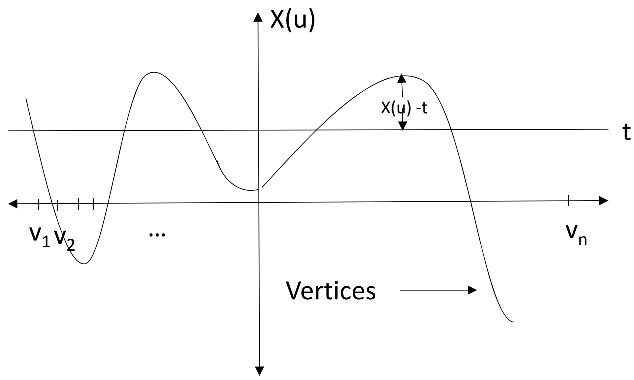

Figure 2: Construction of $\vec{Y}$ and $\vec{Z}$ based on t

Now we prove item 3 of Claim 2. To do that we need to have the following subclaims.

Subclaim 1. $\vec{Y_t}^T L_G \vec{Y_t} + \vec{Z_t}^T L_G \vec{Z_t} \leq \vec{X}^T L_G \vec{X} = \lambda_2$

Proof. One can check:

$$\vec{Y_t}^T L_G \vec{Y_t} + \vec{Z_t}^T L_G \vec{Z_t} = \frac{1}{d} \sum_{u \sim v} \left([\vec{Y_t}(u) - \vec{Y_t}(v)]^2 + [\vec{Z_t}(u) - \vec{Z_t}(v)]^2 \right).$$

This is because, for every pair of u, v, consider two cases for $[Y_t(u) - Y_t(v)]^2 + [Z_t(u) - Z_t(v)]^2$

1. Case 1: $\vec{X}(u), \vec{X}(v)$ both $\geq t$ or $\leq t$. In this case the value of the term will be equal to $[\vec{X}(u) - \vec{X}(v)]^2$
2. Case 2: $\vec{X}(u) > t$ and $\vec{X}(v) < t$ or $\vec{X}(u) < t$ and $\vec{X}(v) > t$. In this case the value of the term will be $< [\vec{X}(u) - \vec{X}(v)]^2$

So

$$\frac{1}{d} \sum_{u \sim v} \left([\vec{Y_t}(u) - \vec{Y_t}(v)]^2 + [\vec{Z_t}(u) - \vec{Z_t}(v)]^2 \right) \leq \frac{1}{d} \sum_{u \sim v} [\vec{X}(u) - \vec{X}(v)]^2 = \vec{X}^T L_G \vec{X}.$$

□

Subclaim 2. $\exists t$ *s.t.* $\left\|\vec{Y}_t\right\|_2^2, \left\|\vec{Z}_t\right\|_2^2 \geq \frac{1}{2}$

Proof. We need to use our assumptions $\vec{X} \perp \vec{1}$ and $\left\|\vec{X}\right\|_2^2 = \vec{X}^T\vec{X} = 1$.

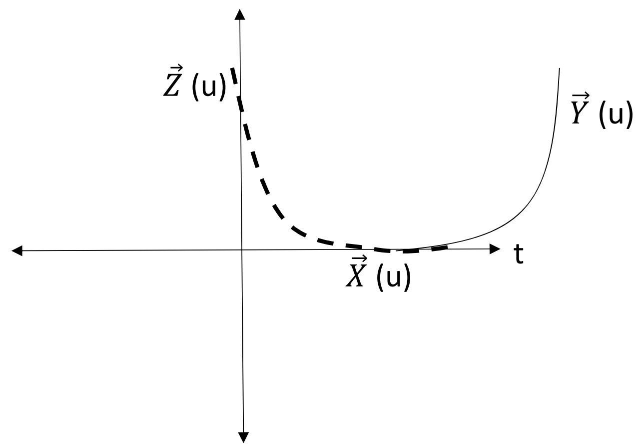

Figure 3: Plot of $\vec{Y}_t$ and $\vec{Z}_t$

As, $\vec{Y}_t(u) = \max\{\vec{X}(u) - t, 0\}$, $\left\|\vec{Y}_t\right\|_2^2 = \sum_{X \in V} \max\{\vec{X}(u) - t, 0\}^2$

Let's try to plot it (Figure 3).

Observation: Both $\vec{Y}_t$ and $\vec{Z}_t$ are continuous functions of t. $\left\|\vec{Y}_t\right\|_2^2$ is continuous and monotonically decreasing. When t tends to ∞, this goes to 0. $\left\|\vec{Z}_t\right\|_2^2$ is also continuous and monotonically increasing. When t tends to ∞, this goes to 0. So both parabolas cross at a point.

So

$$\exists t : \left\|\vec{Y}_t\right\|_2^2 = \left\|\vec{Z}_t\right\|_2^2.$$

Now

$$\left\|\vec{Y}_t\right\|_2^2 + \left\|\vec{Z}_t\right\|_2^2 = \sum_u [\vec{X}(u) - t]^2 = \sum_u \vec{X}(u)^2 - 2t \sum_u \vec{X}(u) + nt^2 = 1 - 0 + nt^2 \geq 1.$$

□

By using Subclaims 1 and 2 we can prove Claim 2, item 3 now:

$$R(L_G, \vec{Y}) = \frac{\vec{Y}^T L_G \vec{Y}}{\vec{Y}^T \vec{Y}} \leq \frac{\lambda_2}{\frac{1}{2}} = 2\lambda_2.$$

Same proof goes for $R(L_G, \vec{Z})$. □

2.2 Step 2

Recall that in Step 2 we want to show that $\exists\, t \geq 0$ s.t. for $S = S_t(\vec{Y})$, $\dfrac{\text{edges}(S, \bar{S})}{d|S|} \leq 2\sqrt{\lambda_2}$.

Proof. In step 1, we have found a convenient $\vec{Y}$ s.t. Rayleigh quotient $\dfrac{\vec{Y}^T L_G \vec{Y}}{\vec{Y}^T \vec{Y}} \leq 2\lambda_2$

Expanding the Rayleigh quotient:

$$2\lambda_2 \geq \frac{\sum_{(u,v)\in E} [\vec{Y}(u) - \vec{Y}(v)]^2}{d\sum_u \vec{Y}^2(u)} = \frac{\left(\sum_{(u,v)\in E} [\vec{Y}(u) - \vec{Y}(v)]^2\right)\left(\sum_{(u,v)\in E} [\vec{Y}(u) + \vec{Y}(v)]^2\right)}{d\sum_u \vec{Y}^2(u)\left(\sum_{(u,v)\in E} [\vec{Y}(u) + \vec{Y}(v)]^2\right)} \tag{1}$$

In the denominator of Equation 1:

$$[\vec{Y}(u) + \vec{Y}(v)]^2 \leq 2[\vec{Y}^2(u) + \vec{Y}^2(v)]$$

$$\implies \sum_{(u,v)\in E} [\vec{Y}(u) + \vec{Y}(v)]^2 \leq 2\sum_{(u,v)\in E} [\vec{Y}^2(u) + \vec{Y}^2(v)]$$

$$= 2d\sum_u \vec{Y}(u)^2, \text{as } G \text{ is a } d\text{-regular graph.}$$

On the enumerator of Equation 1, we use Cauchy-Schwarz inequality.

Theorem 3 (Cauchy-Schwarz)**.** $(\sum_i a_i^2)(\sum_i b_i^2) \geq \sum_i (|a_i b_i|)^2$.

So Equation 1 becomes

$$2\lambda_2 \geq \frac{\sum_{(u,v)\in E} \left(|\vec{Y}^2(u) - \vec{Y}^2(v)|\right)^2}{2\left(d\sum_u \vec{Y}^2(u)\right)^2} \implies \frac{\sum_{u\sim v} \left(|\vec{Y}^2(u) - \vec{Y}^2(v)|\right)}{d\sum_u \vec{Y}^2(u)} \leq 2\sqrt{\lambda_2} \tag{2}$$

The enumerator in Equation 2:

$$\sum_{u\sim v}\left(|\vec{Y}^2(u)-\vec{Y}^2(v)|\right) = \sum_{\substack{u\sim v\\ \vec{Y}^2(u)\le\vec{Y}^2(v)}}\left(\vec{Y}^2(u)-\vec{Y}^2(v)\right)$$
$$= \sum_{\substack{u\sim v\\ \vec{Y}^2(u)\le\vec{Y}^2(v)}}\int_0^\infty 1[\vec{Y}^2(u)\le t<\vec{Y}(v)]\mathrm{d}t.$$

The denominator Equation 2:

$$d\sum_u \vec{Y}^2(u) = d\sum_u\int_0^\infty 1[t<\vec{Y}^2(u)]\mathrm{d}t$$

So Equation 2 becomes:

$$\frac{\sum_{u\sim v}\int_0^\infty 1[\vec{Y}^2(u)\le t<\vec{Y}^2(v)]\mathrm{d}t}{d\sum_u\int_0^\infty [t<\vec{Y}^2(u)]\mathrm{d}t}\le\sqrt{2\lambda_2} \tag{3}$$

Fact 1. *As illustrated in Fig 4, when $a_i, b_i \ge 0$ holds for all $i\in[n]$, we have*

$$min\left\{\frac{b_i}{a_i}\right\}\le\frac{b_1+\cdots+b_n}{a_1+\cdots+a_n}$$

Corollary 4. *If $f(t), g(t)\ge 0$, then,*

$$min\left\{\frac{g(t)}{f(t)}\right\}\le\frac{\int_0^\infty g(t)\mathrm{d}t}{\int_0^\infty f(t)\mathrm{d}t}$$

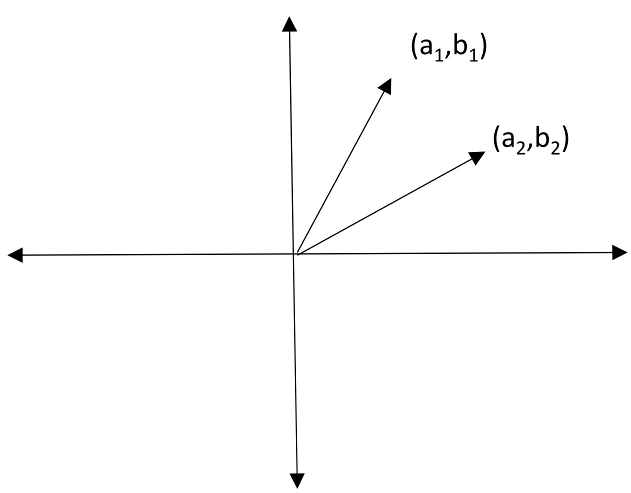

Figure 4: Two vectors with components (a_1, b_1) and (a_2, b_2)

Applying this corollary to Equation 3, we get

$$\exists t \geq 0 : \frac{\sum_{u \sim v} 1[\vec{Y}^2(u) \leq t < \vec{Y}^2(v)]}{d \sum_u 1[t < \vec{Y}^2(u)]} \leq 2\sqrt{\lambda_2}.$$

This implies that

$$\exists t \geq 0 : \frac{\text{edges}(S_{\sqrt{t}}, \bar{S}_{\sqrt{t}})}{d|S_{\sqrt{t}}|} \leq 2\sqrt{\lambda_2}.$$

□