

## Lecture 20: LP Relaxation and Approximation Algorithms

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## 1 Introduction

When variables of constraints of an 0–1 Integer Linear Program (ILP) is replaced by a weaker constraint such that each variable belongs to the interval  $[0, 1]$  (i.e. decimal) instead of fixed values  $\{0, 1\}$  (integers); then this relaxation generates a new Linear Programming problem. For example, the constraint  $x_i \in \{0, 1\}$  becomes  $0 \leq x_i \leq 1$ . This relaxation technique is useful to convert an NP-hard optimization problem (ILP) into a related problem that is solvable in polynomial time. Moreover, the solution to the relaxed linear program can be employed to acquire information about the optimal solution to the original problem. On the other hand, Approximation Algorithms are algorithms used to find approximate solutions to the optimization problems. Linear programming relaxation is an established technique for designing such approximation algorithms for the NP-hard optimization problems (ILP).

**$\alpha$ -approximation:** An algorithm is  $\alpha$ -approximation for any minimization problem if it always outputs a solution with the objective value:  $ALG \leq \alpha \cdot OPT$ , where  $\alpha \geq 1$  and  $OPT$  is the optimal solution for the original problem. For maximization problem (Max-Cut problem), i.e.  $ALG \geq \frac{OPT}{\alpha}$ .

Initially in Section [2], we discuss how to use Linear Programming to approximate the vertex cover problem, which will give such a solution that is ‘comparable’ with the minimum vertex cover. Lastly, we show how to round a Linear Programming relaxation in order to approximate the set cover problem in Section [3].

## 2 Vertex Cover problem

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The problem of finding a minimum vertex cover is a traditional NP-hard optimization problem that has an approximation algorithm. Its decision version is called the vertex cover problem.

### Formulation of vertex cover

- Input: undirected graph  $G = (V, E)$
- Goal: Find  $S \subseteq V$  such that
  - 1)  $\forall \{i, j\} \in E, \{i, j\} \cap S \neq \emptyset$
  - 2)  $|S|$  is minimized

Now from the recent definition of  $\alpha$ -approximation, 2-approximation for vertex cover means that the solution (vertex cover) it produces is at most 2 times of the smallest vertex cover of the given graph, this is guaranteed.

### Formulation of ILP version of vertex cover:

Let

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

$$\begin{aligned} & \text{minimize} && \sum_{i \in V} x_i \\ & \text{subject to} && x_i \in \{0, 1\}, \forall i \in V \\ & && x_i + x_j \geq 1, \forall \{i, j\} \in E. \end{aligned}$$

This is still NP-hard for the presence of integral constraint of  $x_i \in \{0, 1\}$ . Now we can relax the integral constraint by  $x_i \in [0, 1]$ , which introduces the following

### Formulation of LP Relaxation version of vertex cover:

$$\begin{aligned} & \text{minimize} && \sum_{i \in V} x_i \\ & \text{subject to} && 0 \leq x_i \leq 1, \forall i \in V \\ & && x_i + x_j \geq 1, \forall \{i, j\} \in E. \end{aligned}$$

This is LP and solvable in polynomial time. Each solution for ILP remains valid for LP because of the relaxation procedure.

**Claim 1.**  $\text{OPT}_{LP} \leq \text{OPT}_{ILP}$ , for a minimization problem (Vertex Cover).

LP shouldn't be arbitrarily smaller rather lower bounded so that LP can work as some kind of estimator for ILP. Now we discuss about the extra solutions (fractional) that are introduced specially for the relaxation. For example, if a graph has three vertices and each are connected, then one solution of the ILP problem is  $(1, 1, 0)$  where  $\text{OPT}_{ILP} = 2$ . From the LP relaxation if two solutions arise like  $(0.7, 0.7, 0.3)$  or  $(0.5, 0.5, 0.5)$ ; then  $\text{OPT}_{LP}$  is 1.7 or 1.5 respectively. It shows  $\text{OPT}_{LP} \leq \text{OPT}_{ILP}$ .

**Rounding procedure:** This process converts fractional solutions to integral solutions, its true that it may loose some objective value, which corresponds to the approximation factor. The following rounding technique for vertex cover problem is straightforward–

The rounding solution

$$x_i^* = \begin{cases} 1 & \text{if } x_i \geq \frac{1}{2} \\ 0 & \text{if } x_i < \frac{1}{2} \end{cases}$$

By this rounding procedure our toy example from above, the fractional solution turns to integral solution like (1, 1, 0) and (1, 1, 1) where objective values are 2 and 3 respectively.

**Claim 2.** Round  $\leq 2 \cdot \text{OPT}_{ILP}$ .

*Proof.*

$$\begin{aligned} \sum_{i \in V} x_i^* &\leq \sum_{i \in V} 2x_i = 2 \sum_{i \in V} x_i = 2 \cdot \text{OPT}_{LP} \leq 2 \cdot \text{OPT}_{ILP} \\ &\implies \text{Round} \leq 2 \cdot \text{OPT}_{ILP}. \end{aligned}$$

□

From this proof, we have the following immediate claim–

**Claim 3.** Solving LP relaxation using rounding technique is 2-approximation for Vertex Cover.

**Claim 4.** LP relaxation using rounding technique returns a feasible solution.

*Proof.* Since  $\forall(i, j) x_i + x_j \geq 1$ , hence there exists at least one vertex which is greater than  $\frac{1}{2}$ ; i.e.,  $x_i \geq \frac{1}{2} \implies x_i^* = 1$ . Which guarantees the feasible solution existence. □

## 2.1 Integrality gaps

From the last three claims, we have the following corollary–

**Corollary 5.**  $\text{OPT}_{ILP} \leq \text{Round} \leq 2 \cdot \text{OPT}_{LP}$  and  $\text{OPT}_{ILP} \in [\text{OPT}_{LP}, 2 \cdot \text{OPT}_{LP}]$ .

Now we introduce an important concept like integrality gap to demonstrate the best approximation guarantee can be achieved by using the LP relaxation as a lower bound. For a minimization problem, it is the worst case ratio, i.e.  $\text{IG} = \frac{\text{OPT}_{ILP}}{\text{OPT}_{LP}}$ . Which indicates that the LP relaxation with rounding Vertex Cover has  $\text{IG} \in [1, 2]$ . This tells that if we have an instance of a undirected graph, then we couldn't give the optimal vertex cover solution rather an estimate of the vertex cover with factor 2. So, as we said earlier, LP is an estimator of ILP.

At this point, we have questions like how we can improve this estimator. The answer can be given in two ways,

1. we can use ‘different LP’
2. we can perform ‘better analysis’.

Firstly, we describe the second option– better analysis for LP. We have the following observation–

**Observation 6.** *Can’t beat the factor of  $\frac{4}{3}$ .*

For example, let’s assume for a triangular graph (three vertices),  $\text{OPT}_{LP} = 1.5$  and  $\text{OPT}_{ILP} = 2$ , then the IG is already  $\frac{4}{3}$ .

For better Integrality Gap we consider the complete graph  $K_n$  with  $n$  vertices. Now if a solution (feasible) for the LP Vertex Cover problem sets  $x_i = \frac{1}{2}, \forall i$ , then  $\text{OPT}_{LP} \leq \frac{n}{2}$ . It’s trivial that for full connectivity, the smallest vertex cover gives  $\text{OPT}_{ILP} = n-1$ . Hence,  $\text{IG} = 2 - \frac{2}{n} = 2 - o(\frac{1}{n})$ , so this is such an instance where LP is off by factor of 2. Asymptotically, it becomes exactly 2 when  $n \rightarrow \infty$ . We can say that  $K_n$  is a 2-integrality gap instance for LP and a certificate on the bad estimator of LP.

This observations help us to understand the NP-hardness of the problem. It cannot be approximated arbitrarily well unless  $P = NP$ . The theorem of Dinur et al. [2] shows that computing a vertex cover is within a factor of 1.36 of the smallest vertex cover is as hard as computing the smallest vertex cover itself; hence the approximation ratio is also 1.36. We have another observation from Knot and Regev– if we assume Unique Games Conjecture holds, then it cannot be approximated within any constant factor better than  $2 - \epsilon$ . Furthermore, the current best approximation ratio for Vertex Cover is  $2 - \theta(\frac{1}{\sqrt{\log n}})$  shown by Karakostas [1].

### 3 Set Cover Problem

Given a set of elements  $\{1, 2, \dots, n\}$  (called the universe,  $U$ ) and a collection of  $m$  sets whose union equals the universe, the set cover problem is to identify the smallest sub-collection of the sets whose union equals the universe.

#### Formulation of set cover

- Input: Universe,  $U = [n] = \{1, 2, \dots, n\}$ ; Collection  $S_1, S_2, \dots, S_m \subseteq U$ .
- Goal: Find smallest  $I \subseteq \{1, 2, \dots, m\}$  such that  $\bigcup_{i \in I} S_i = U$

**Formulation of ILP version of set cover:**

Let

$$x_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$$

$$\begin{aligned} &\text{minimize} && \sum_{i \in [m]} x_i \\ &\text{subject to} && x_i \in \{0, 1\}, \forall i \in [m] \\ &&& \sum_{i: u \in S_i} x_i \geq 1, \forall u \in U. \end{aligned}$$

**Formulation of LP relaxation version of set cover:**

$$\begin{aligned} &\text{minimize} && \sum_{i \in [m]} x_i \\ &\text{subject to} && 0 \leq x_i \leq 1, \forall i \in [m] \\ &&& \sum_{i: u \in S_i} x_i \geq 1, \forall u \in U. \end{aligned}$$

**Rounding technique:** This LP relaxation gives optimal fractional solution where  $x_i^* \in [0, 1]$  for each set  $S_i$ . However, we cannot apply rounding technique alike previous vertex cover problem here. Because an element can remain uncovered when it belongs to many sets with  $x_i^* \sim 0$  for each of them. In this case, if we know that an element can belong to at most  $k$  number of sets, then we can use  $\frac{1}{k}$  as a threshold to round the fractional solutions to integral solutions; like–

$$x_i^* = \begin{cases} 1 & \text{if } x_i \geq \frac{1}{k} \\ 0 & \text{if } x_i < \frac{1}{k} \end{cases}$$

Note that it is a feasible cover being  $k$ -approximation. However, in worst case  $k$  could be really large like  $\frac{n}{2}$ . So we are trying to find a better approximation guarantee.

**RandomPick technique:** We consider here each  $x_i^*$  as a probability and  $x^*$  feasible solution as a probability distribution on options for choosing the subsets, e.g.  $x_1^*$  is the probability to select  $S_1$  set. The algorithm works as follows:

- Input: fractional feasible solution from LP relaxation,  $x_1^*, \dots, x_m^*$ .
- $I = \phi$
- for  $i = 1$  to  $n$

$$I = I \cup i, \text{ w.p. } x_i^*.$$

- Return  $I$ .

The following claim is trivial, i.e. the expected cost of the sets we can pick is exactly the objective function of the LP relaxation and bounded by ILP.

**Claim 7.**  $E(|I|) = \sum_{i \in [m]} x_i = \text{OPT}_{LP} \leq \text{OPT}_{ILP}$ .

Still, there is high probability such that an element could remain uncovered and the probability an element will be covered is not exactly 1. We first investigate what is the probability that an element is covered.

**Claim 8.**  $\Pr[u \in \bigcup_{i \in I} S_i] \geq 1 - \frac{1}{e}$ .

*Proof.* We first figure out  $\Pr[u \notin \bigcup_{i \in I} S_i]$ , i.e. probability that an element could not be covered.

$$\Pr[u \notin \bigcup_{i \in I} S_i] = \prod_{i: u \in S_i} \Pr[i \in I] = \prod_{i: u \in S_i} (1 - x_i^*) \leq \prod_{i: u \in S_i} e^{-x_i^*}$$

We know the fact that  $1 - t \leq e^{-t}$ ,  $\forall t \geq 0$  from Taylor series expansion of  $e^{-t}$  and from the constraint we have  $\sum_{i: u \in S_i} x_i \geq 1$ . So–

$$\Pr[u \notin \bigcup_{i \in I} S_i] \leq e^{-\sum_{i: u \in S_i} x_i^*} \leq e^{-1} = \frac{1}{e}.$$

Eventually,

$$\Pr[u \in \bigcup_{i \in I} S_i] \geq 1 - \frac{1}{e}$$

□

**Remark 1.** LP relaxation using the RandomPick technique returns a set  $I$  that covers any element w.p.  $\geq 1 - \frac{1}{e}$ .

**RandomizedRound technique:** To confirm that no element remains uncovered, we iterate RandomPick technique  $\lceil 2 \ln n \rceil$  times. The algorithm is as follows–

- Iterate  $\lceil 2 \ln n \rceil$  times.  
At each iteration  $j$ ,  $I_j \leftarrow \text{RandomPick}$
- Return  $I = \bigcup_{j=1}^{\lceil 2 \ln n \rceil} I_j$ .

**Remark 2.**  $E[|I|] \leq \sum_j E[|I_j|] \leq \lceil 2 \ln n \rceil \cdot \sum_{i \in [m]} x_i = \lceil 2 \ln n \rceil \cdot \text{OPT}_{ILP}$

**Claim 9.**  $\Pr[u \in \bigcup_{i \in I} S_i] \geq 1 - \frac{1}{n^2}$ .

*Proof.*

$$\Pr[u \in \bigcup_{i \in I} S_i] = 1 - \prod_{j=1}^{\lceil 2 \ln n \rceil} \Pr[u \notin \bigcup_{i \in I_j} S_i] \geq 1 - (e^{-1})^{\lceil 2 \ln n \rceil} \geq 1 - \frac{1}{n^2}.$$

□

Now we figure out how probable does  $I$  cover the full universe  $U$  by the following corollary.

**Corollary 10.**  $\Pr[I \text{ covers } U] \geq 1 - \frac{1}{n}$ .

*Proof.* For any  $u \in U$ , we already saw,  $\Pr[u \notin \bigcup_{i \in I_j} S_i] \leq \frac{1}{n^2}$ .

Now for the whole universe we apply union bound on these to get

$$\Pr[\exists u \in U : u \text{ not covered}] = \frac{1}{n^2} \cdot |U| = \frac{1}{n^2} \cdot n = \frac{1}{n}.$$

Finally,

$$\Pr[I \text{ covers } U] \geq 1 - \frac{1}{n}.$$

□

To observe that LP relaxation with RandomizedRound techniques gives us  $O(\ln n)$ -approximation algorithm, we have the following theorem—

**Theorem 11.** RandomizedRound returns  $I$  s.t.  $I$  is a set cover for  $U$  and  $|I| \leq (4 \ln n + 2) \cdot \text{OPT}_{ILP}$  with probability  $\frac{1}{2} - \frac{1}{n} \geq 0.4$  (for large enough  $n$ ).

*Proof.* At first we evaluate with the help of Markov bound,

$$\begin{aligned} & \Pr[|I| \leq (4 \ln n + 2) \cdot \text{OPT}_{ILP}] \\ &= 1 - \Pr[|I| > (4 \ln n + 2) \cdot \text{OPT}_{ILP}] \\ &\geq 1 - \frac{E[|I|]}{(4 \ln n + 2) \cdot \text{OPT}_{ILP}} \\ &\geq 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

By Corollary 10 we already know  $\Pr[I \text{ covers } U] \geq 1 - \frac{1}{n}$ . Hence, by union bound—

$$\begin{aligned} & \Pr[|I| \leq (4 \ln n + 2) \cdot \text{OPT}_{ILP} \ \& \ I \text{ covers } U] \\ &\geq 1 - (1 - \frac{1}{2}) - (1 - (1 - \frac{1}{n})) = \frac{1}{2} - \frac{1}{n} \end{aligned}$$

□

## References

- [1] G. Karakostas. A better approximation ratio for the Vertex Cover problem. ECCC Report TR04-084, 2004.
  
- [2] I. Dinur and S. Safra. The importance of being biased. Proceedings of the 34th Annual ACM Symposium on Theory of Computing, pages 33-42, 2002.