CSCI-B609: A Theorist's Toolkit, Fall 2016 Nov 17

Lecture 22: Hyperplane Rounding for Max-Cut SDP

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1 Introduction

In the previous lectures we had seen the max-cut problem. In this lecture we will present a 0.878-approximation algorithm. We will in fact, also see that we do not have much hope of doing better.

2 Recap

Let us have quick recap of the semi-definite Programming for the max-cut problem. Figure 1 is the Qudratic Program for the Max-Cut. Figure 3 is the relaxation for the SDP.

Fact 1. *Since* SDP *is a relaxation of the* QIP*,* SDP $\geq$ QIP*.*

We can think of SDP as a relaxation of the Integer Program, because if we add an extra constraint rank($Y = 1$), it will become a Quadratic Integer Program. We know from the previous lecture that,

Theorem 1. SDP *is solvable in polynomial time using ellipsoid method and a separation oracle.*

3 Vector View

Recall Choleskey Decomposition. Given $Y \succeq 0$, we can write Y as $Y = L^T DL$ where L is

Corollary 2. *We can also write* $Y = L'^T L'$ *where* $L' = \sqrt{D}L$*.*

In fact, Y contains inner product of two vectors. $y_{uv} = \langle \mathbf{w}_u, \mathbf{w}_v \rangle$. Write $L' = [\mathbf{W}_1, \mathbf{W}_1, \cdots, \mathbf{W}_n]$, We have $Y_{uv} = \mathbf{w}_u^T \mathbf{w}_v$. Finally the SDP can be written as it is in Figure 3.

This Integer Program can be relaxed in the following manner.

$$\begin{aligned} \text{maximize: } & \sum_{(u,v)\in E} \frac{1 - x_u x_v}{2} \\ \text{subject to: } & x_u^2 = 1 \ \forall u \in V \end{aligned}$$

Figure 1: Quadratic Integer Program for Max-Cut

$$\begin{aligned} \text{maximize: } & \sum_{(u,v)\in E} \frac{1 - y_{uv}}{2} \\ \text{subject to: } & y_{uv} = y_{vu} \ \forall u, v \in V \\ & y_{uu} = 1 \ \forall u \in V \\ & Y \succeq 1 \end{aligned}$$

Figure 2: SDP for Max-Cut

$$\begin{aligned} \text{maximize: } & \sum_{(u,v)\in E} \frac{1 - \langle \mathbf{w}_u, \mathbf{w}_v \rangle}{2} \\ \text{subject to: } & ||w_u||_2^2 = 1 \ \forall u \in V \end{aligned}$$

Figure 3: SDP for Max-Cut

3.1 Hyperplane Rounding

The main idea is to use a random hyperplane that goes through the origin to divide the vectors in to two sets, corresponding to a cut. It can summarized in the following steps.

1. Choose a uniform random hyperplane through the origin that divides the sphere. In other words, Choose a norm-vector with a uniform random direction; sample $\mathbf{g} = (\mathbf{g}_1, \cdots, \mathbf{g}_n) \sim \mathcal{N}(0,1)^n$.
2. Set $\mathbf{x}_u = \text{sign}(\mathbf{g}, \mathbf{w}_u)\ \forall u \in V$

Theorem 3. *[GW95]* $\mathrm{E}[\sum_{(u,v)\in E} \frac{1-x_u x_v}{2}] \geq 0.878(\text{SDP})$

Proof. By linearity of expectation.

$$\text{LHS} = \sum_{(u,v)\in E} \mathrm{E}\left[\frac{1-x_u x_v}{2}\right]$$
$$= \sum_{(u,v)\in E} \Pr[\mathbf{w}_u, \mathbf{w}_v \text{ separated by a random hyperplane}]$$

Now for a pair of fixed u and v, let us focus on the plane containing $\mathbf{w}_u$ and $\mathbf{w}_v$. By symmetry the random hyperplane's projection becomes a random line through the origin. Thus we have the following.

$$\Pr[\mathbf{w}_v \text{ and } \mathbf{w}_u \text{ are separated}] = \frac{\angle(\mathbf{w}_v, \mathbf{w}_u)}{\pi}$$
$$= \frac{\arccos(\mathbf{w}_v, \mathbf{w}_u)}{\pi}$$
$$\implies \sum_{(u,v)\in E} \Pr[\mathbf{w}_u, \mathbf{w}_v \text{ separated by a random hyperplane}] = \sum_{(u,v)\in E} \frac{\arccos(\mathbf{w}_v, \mathbf{w}_u)}{\pi}$$

Now let $\alpha_{GW} = \min_{\rho\in[-1,1]}\{\frac{(\arccos\rho)/\pi}{(1-\rho)/2}\}$. Using this we have

$$\sum_{(u,v)\in E} \frac{\arccos(\mathbf{w}_v, \mathbf{w}_u)}{\pi} \geq \sum_{(u,v)\in E} \alpha_{GW}.\frac{1-\langle \mathbf{w}_v, \mathbf{w}_u\rangle}{2} = \alpha_{GW}.(\text{SDP})$$

Numerical results show that, $\alpha_{GW} \sim 0.87856\cdots > 0.878$ □

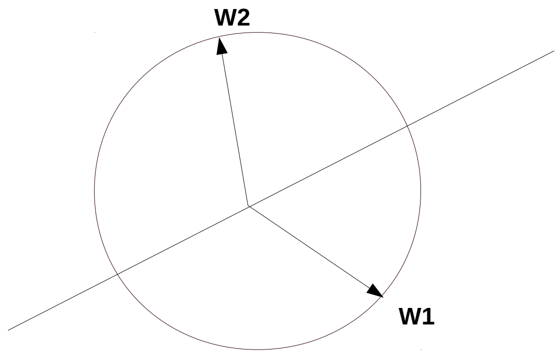

Figure 4: Hyperplane Rounding

4 Recognizing almost Bipartite Graphs

Suppose OPT $\geq (1-\epsilon)m$. m is the number of edges ($|E|$). The graph is bipartite after removing ϵm edges. Recall that, $\mathrm{E}[\text{rounding}] \geq (1-\epsilon)0.878m$. The question we ask here is can we do better than this?

Intuition: When $\epsilon = 0$, $\exists$ poly-time algorithm: returns a cut with m edges (bipartite graph recognition). Let alg(c) be the best found by a polytime algorithm when OPT $= c.m$. The curve of alg(c) is not continuous at $c = 1$. Is this really the case? *GW* finds a cut of size $(1 - O(\sqrt{\epsilon}))m$ given OPT $= (1-\epsilon)m$. Now we can state the following theorem.

Theorem 4. $\text{alg}(1-\epsilon) \geq O(\sqrt{\epsilon})$

Observation: $\cos x = 1 - \frac{x^2}{2} + O(x^4) \implies \arccos(1-y) = \sqrt{2y} + O(y)$.

Claim 1. $\arccos(1-y) \leq 4\sqrt{y}\ \forall y \in [0,2]$

Proof.

$$\begin{aligned}
&\text{at } y = 0 : \text{LHS} = \text{RHS} = 0 \\
&\text{at } y \in (0,1) : \frac{d\text{LHS}}{dy} = \frac{1}{\sqrt{y(2-y)}} \leq \frac{1}{\sqrt{y}} \leq \frac{1}{2\sqrt{y}} = \frac{d\text{RHS}}{dy} \\
&\text{at } y \in [1,2]\text{: LHS} \leq \arccos(-1) = \pi \\
&\text{RHS} \geq 4
\end{aligned}$$

□

Corollary 5. $\arccos(1-y) = \pi - \arccos(1-y) \geq \pi - 4\sqrt{y}$

Theorem 6. *Assuming the Unique Games Conjecture, there exists no poly-time algorithm that* $(\alpha_{GW} + \delta)$ *or* $(1-\epsilon, 1-o(\sqrt{\epsilon})$*-approximate max-cut for all* $\epsilon, \delta > 0$.

Proof.

$$\begin{aligned}
\mathrm{E}[\text{rounding}] &= \sum_{(u,v)\in E} \Pr[\mathbf{w}_u, \mathbf{w}_v \text{ separated}] \\
&= \sum_{(u,v)\in E} \frac{\arccos(\langle \mathbf{w}_u, \mathbf{w}_v \rangle)}{\pi} \\
&= \sum_{(u,v)\in E} \frac{\arccos(\epsilon_{uv} - 1)}{\pi} \\
&\geq \sum_{(u,v)\in E} \frac{\pi - 4\sqrt{\epsilon_{uv}}}{\pi} \\
&\geq m - \frac{4}{\pi} \sum_{(u,v)\in E} \sqrt{\epsilon_{uv}} \ (*)
\end{aligned}$$

Using Jensen's inequality we have $(*) \geq m - \frac{4}{\pi} m \sqrt{\frac{1}{m} \sum_{(u,v)\in E} \epsilon_{uv}}$ $(**)$

$$\begin{aligned}
\sum_{(u,v)\in E} \epsilon_{uv} &= m + \sum_{(u,v)\in E} \langle \mathbf{w}_u, \mathbf{w}_v \rangle \\
&= m + (m - 2\mathrm{OPT}_{\mathrm{SDP}}) \\
&\leq 2(m - \mathrm{OPT}_{\mathrm{QIP}}) \\
&\leq 2\epsilon \\
\text{Therefore, } (**) &\geq m - \frac{4}{\pi} m \sqrt{2\epsilon}
\end{aligned}$$

□

5 Constraint Satisfaction Problem

In this section we talk about the Constraint Satisfaction Problems (CSP).
Domain: $\Omega = \{1, 2, \cdots, q\}$
Predicates: $\Pi = \{\pi : \Omega^k \to \{0, 1\}\}$
Input: n variables $x_1, \cdots, x_n$, m constraints in the form $(x_{i_1}, \cdots, x_{i_k}, \pi \in \Pi)$
Goal: Find assignment $\sigma : \{x_1, \cdots, x_n\} \to \Omega$ to maximize number of satisfied constraints.
A constraint is satisfied if and only if $\pi(\sigma(x_{i_1}), \cdots, \sigma(x_{i_k})) = 1$

$$
\begin{aligned}
\text{maximize: } & \sum_{a,b\in\Omega:\pi(a,b)=1} \langle \mathbf{w}_{ia}, \mathbf{w}_{jb} \rangle \\
\text{subject to: } & \langle \mathbf{w}_{i,a}, \mathbf{w}_{i,b} \rangle = 0 \ \forall i \in [n], a, b \in \Omega, a \neq b \\
& \langle \mathbf{w}_{i,a}, \mathbf{w}_{i,b} \rangle \geq 0 \ \forall i, j \in [n], a, b \in \Omega \\
& \sum_{a\in\omega} \mathbf{w}_{i,a} = \mathbf{I} \ \forall i \in [n] \\
& ||\mathbf{I}||_2^2 = 1
\end{aligned}
$$

Figure 5: Basic SDP

5.1 Basic SDP Relaxation for CSP

Here we will deal with $k = 2$, (binary CSP). For each x_i and $a \in \omega$ introduce $\mathbf{w}_{i,a}$ correspond to $\sigma(x_i) = a$. Note that in the integral solution $\mathbf{w}_{i,a} = \mathbf{I}$, if $\sigma(x_i) = a$ and $\mathbf{w}_{i,a} = 0$ otherwise. Thus we get the **Basic SDP** as in Figure 5.

Theorem 7. *[Rag08] For every CSP, $\exists$ polynomial time rounding scheme for BasicSDP achieving the optimal approximation guarantee among all ploy-time algorithms assuming the Unique Games Conjecture.*

References

[GW95] Michel X. Goemans and David P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42(6):1115–1145, November 1995.

[Rag08] Prasad Raghavendra. Optimal algorithms and inapproximability results for every csp? In *Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing*, STOC '08, pages 245–254, New York, NY, USA, 2008. ACM.