Boolean Functions II

David Lu

Review: <u>Boolean function</u> is $f: \{-1, 1\}^n \to \{-1, 1\}$

Basically, assigning ± 1 to each vertex of n-cube

We can model a function f as a polynomial!
Note that $\left(\frac{1}{2}+\frac{x_1}{2}\right)\left(\frac{1}{2}+\frac{x_2}{2}\right)\left(\frac{1}{2}+\frac{x_3}{2}\right)$ only has nonzero value at $(1,1,1)$. So (assume $n=3$ for simplicity)

$$f = \left(\frac{1}{2}+\frac{x_1}{2}\right)\left(\frac{1}{2}+\frac{x_2}{2}\right)\left(\frac{1}{2}+\frac{x_3}{2}\right) f(1,1,1)$$
$$+ \left(\frac{1}{2}+\frac{x_1}{2}\right)\left(\frac{1}{2}+\frac{x_2}{2}\right)\left(-\frac{1}{2}+\frac{x_3}{2}\right) f(1,1,-1)$$
$$+ \cdots$$
$$= [\text{some multilinear (all exponents are 0 or 1) poly.}]$$

We define <u>$\hat{f}(S)$</u> for $S \subseteq [n]$ to be the coefficient of $\prod_{i \in S} x^i$ in this <u>Fourier expansion</u>.

E.g. $\hat{f}(\{1,2,3\}) = 3$ for $f = 3x_1x_2x_3 + 2x_1x_2 + 5x_3 + 4x_2x_3$.

<u>Inner Product</u>: $\langle f, g \rangle = \mathop{E}_{x \sim \{-1,1\}^n}\left[f(x)\, g(x)\right]$

(also equal to $1 - 2\Pr_x\left[f(x) \neq g(x)\right]$)

Parseval's Theorem: For f and g,

$$\langle f, g \rangle = \sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S)$$

(end of review)

Influence: "Probability that individual bit affects outcome"

Formally: We call coordinate $i \in [n]$ pivotal for fn. f and input x if $f(x) \neq f(x^{\oplus i})$

$x^{\oplus i} = (x_1, x_2, \cdots x_{i-1}, -x_i, x_{i+1}, \cdots)$ (flip the bit with index i)

The influence of coordinate i on $f: \{-1, 1\}^n \to \{-1, 1\}$ is:

$$\mathrm{Inf}_i[f] = \Pr_{x \sim \{-1,1\}^n} [f(x) \neq f(x^{\oplus i})]$$

(probability that i is pivotal for a random input)

EXAMPLES ON PG. 4

We claim that this also equals $\sum_{\substack{S \subseteq [n] \\ S \ni i}} \hat{f}(S)^2$.

For simplicity, let $f(x^{\oplus i}) = g(x)$.

Note that $\Pr[f(x) \neq g(x)]$ is equivalent to

$$\frac{1}{4} E\left((f(x) - g(x))^2 \right)$$

$$= \frac{1}{4} E\left(f(x)^2 + g(x)^2 - 2 f(x) g(x) \right)$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{1}{2} E(f(x) g(x)) \quad \text{since } E(f(x)^2) = 1$$

$$= \frac{1}{2} - \frac{1}{2}\langle f, g\rangle \qquad \text{since } \langle f,g\rangle = E(f(x)g(x))$$

$$= \frac{1}{2}\langle f, f\rangle - \frac{1}{2}\langle f, g\rangle \qquad \text{since } 1 = E(f(x)^2) = \langle f, f\rangle$$

$$= \frac{1}{2}\sum_{S\subseteq[n]} \hat{f}(S)^2 - \frac{1}{2}\sum_{S\subseteq[n]} \hat{f}(S)\,\hat{g}(S) \qquad \text{by Parseval's}$$

$$= \frac{1}{2}\sum_{S\not\ni i} \hat{f}(S)^2 + \frac{1}{2}\sum_{S\ni i} \hat{f}(S)^2 - \frac{1}{2}\sum_{S\not\ni i} \hat{f}(S)\,\hat{g}(S) - \frac{1}{2}\sum_{S\ni i} \hat{f}(S)\,\hat{g}(S)$$

Note that if $i \notin S$, $\hat{f}(S) = \hat{g}(S)$ and if $i \in S$, $\hat{f}(S) = -\hat{g}(S)$. This is because $g = f^{\oplus i}$, meaning the Fourier expansion of g is exactly that of f, except with all (x_i)'s turned into $(-x_i)$'s. So:

$$= \frac{1}{2}\sum_{S\not\ni i} \hat{f}(S)^2 + \frac{1}{2}\sum_{S\ni i} \hat{f}(S)^2 - \frac{1}{2}\sum_{S\not\ni i} \hat{f}(S)^2 + \frac{1}{2}\sum_{S\ni i} \hat{f}(S)^2$$

$$= \sum_{S\ni i} \hat{f}(S)^2.$$

$\blacksquare$

Influence examples

- Dictator function: $f(x_1 x_2 x_3 \dots x_n) = x_d$ for some d

What is $\mathrm{Inf}_i[f]$ for $i \neq d$? ... $\Pr[f(x) \neq f(x^{\oplus i})]$

Since $f(x) = x_d$ this probability is 0.

Similarly, $\mathrm{Inf}_i[f] = 1$, since $f(x)$ always $\neq f(x^{\oplus d})$.

- Parity function: $f(x_1 x_2 x_3 \dots x_n) = \prod_i x_i$

What is $\mathrm{Inf}_i[f]$?

Changing bit i will <u>always</u> flip the value of f, so

$\mathrm{Inf}(i) = 1$ for all i.

- Majority function: $f(x_1 x_2 \dots x_n) = \begin{cases} +1 & \text{if majority } x_i = +1 \\ -1 & \text{otherwise} \end{cases}$

(n is odd)

Changing bit i will flip the value of f only if the other bits are split evenly between ± 1.

For a random x, this has probability $\dfrac{\binom{n-1}{n-1/2}}{2^{n-1}}$.

Stirling's formula: $m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}$

$$\Rightarrow \mathrm{Inf}_i[f] \sim \frac{1}{2^{n-1}} \frac{\left(\frac{n-1}{e}\right)^{n-1} \sqrt{2\pi(n-1)}}{\left(\frac{n-1}{2e}\right)^{n-1} 2\pi \frac{n-1}{2}} = \frac{1}{2^{n-1}} 2^{n-1} \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{n-1}}$$

$$= O\left(\frac{1}{\sqrt{n}}\right)$$

Noise Stability: "Probability that noise in the bits affects the output of the function"

Formally: Let $\rho \in [-1, 1]$. For input $x \in \{-1, 1\}^n$, let y be a string drawn such that:

$$y_i = \begin{cases} x_i & \text{with probability } \frac{1}{2} + \frac{1}{2}\rho \\ -x_i & \text{with probability } \frac{1}{2} - \frac{1}{2}\rho \end{cases}$$

We say that y is <u>ρ-correlated</u> to x.

The <u>noise stability</u> of f at ρ is:

$$\mathrm{Stab}_\rho[f] = \mathop{E}_{\substack{(x,y) \\ \rho\text{-correlated}}} \left[f(x) f(y) \right] = 2 \cdot \Pr\left[f(x) = f(y) \right] - 1$$
$$= 1 - 2 \cdot \Pr\left[f(x) \neq f(y) \right]$$

- Dictator function: $f(x_1 x_2 \ldots x_n) = x_i$ for some i

  $f(y) \neq f(x)$ only if $y_i \neq x_i$, since no other y_j matters. $\Pr[y_i \neq x_i] = \left(\frac{1}{2} - \frac{1}{2}\rho\right)$

  $\Rightarrow \mathrm{Stab}_\rho[f] = 1 - 2\left(\frac{1}{2} - \frac{1}{2}\rho\right) = \rho$

- Majority function: harder!

$$\mathrm{Stab}_\rho[\mathrm{Maj}_n] = E[\mathrm{Maj}_n x \cdot \mathrm{Maj}_n y]$$
$$= E\left[\mathrm{sign}\left(\sum \tfrac{1}{\sqrt{n}} x_i\right) \cdot \mathrm{sign}\left(\sum \tfrac{1}{\sqrt{n}} y_i\right) \right]$$
$$= 1 - 2 \cdot \Pr\left[\mathrm{sign}\left(\sum \tfrac{1}{\sqrt{n}} x_i\right) \neq \mathrm{sign}\left(\sum \tfrac{1}{\sqrt{n}} y_i\right) \right]$$

For $n \to \infty$, $\sum \frac{1}{\sqrt{n}} x_i \to g_1$ and $\sum \frac{1}{\sqrt{n}} y_i \to g_2$ where g_1, g_2 are ρ-correlated Gaussians.

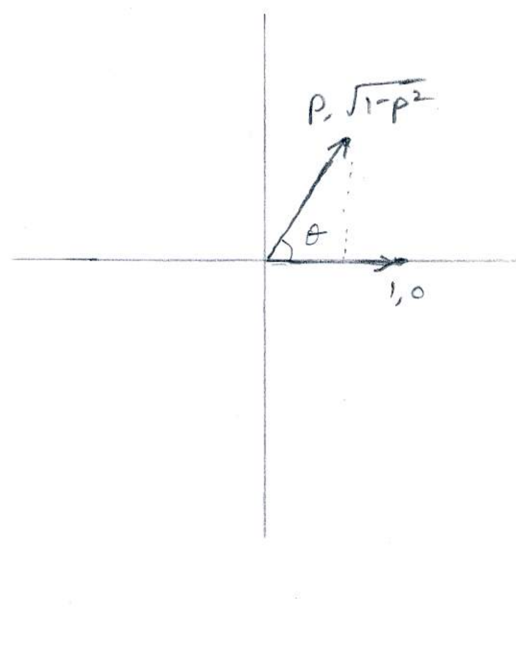
$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right) \quad \text{(mean, cov. matrix)}$$

i.e. $E[g_1] = E[g_2] = 0$, $E[g_1^2] = E[g_2^2] = 1$, and $E[g_1 g_2] = \rho$

This is equivalent to $g_1 = z_1$, $g_2 = \rho z_1 + \sqrt{1-\rho^2}\, z_2$ for <u>independent</u> $z_1, z_2 \sim N(0, 1)$. You can verify that all expected values check out!

So $g_1 = (1, 0) \cdot (z_1, z_2)$ and $g_2 = (\rho, \sqrt{1-\rho^2}) \cdot (z_1, z_2)$.

g_1 and g_2 will have opposite signs iff the normal vector to (z_1, z_2) cuts through the angle θ.



The angle of (z_1, z_2) is uniformly random, and θ here equals $\arccos(\rho)$, so the probability is $\frac{1}{\pi} \arccos(\rho)$. This gives us:

$$\text{Stab}_\rho[\text{Maj}_n] = 1 - \frac{2}{\pi} \arccos \rho$$

$$= \frac{2}{\pi} \arcsin \rho$$

Majority is Stablest: For $0 < p < 1$ and $f : \{-1, 1\}^n \to \{-1, 1\}$, if:

1. $E[f(x)] = 0$ for random x, and
2. $Inf_i[f] \leq \epsilon$ for any $i \in [n]$, then

- $Stab_p[f] \leq \frac{2}{\pi} \arcsin(p) + O\left(\frac{\log \log \frac{1}{\epsilon}}{\log \epsilon}\right)$

So $Maj_p[f]$ is the 'stablest' function!

Note that conditions 1) and 2) are necessary.

- If 1) is taken away, the constant function $f(x) = 1$ gives trivial 1 stability.
- If 2) is taken away, the dictator function $f(x) = x_i$ gives easy p stability.