Expanders II - Applications

Expander codes

'Good code': Positive constant rate
Positive constant min. distance
~ Efficient to decode / encode

Theorem (Margulis)

For $d \gtrsim 64$, there exists a left d-regular bipartite graph with $|L| = n$ and $|R| = \frac{3}{4}n$, s.t. $|N(S)| \geq 0.8\, d\, |S|$ for all $S \subseteq L : |S| \leq \frac{0.02}{d} n$

- Explicit construction!

TANNER CODE (a type of linear code)

$$\begin{array}{cc} & |L| \\ |R| & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & \cdot & \cdot & \\ \vdots & & & & \end{bmatrix} \end{array}$$

Take $\underline{d = 64}$ and the Margulis expander.

Our code is defined using the parity check matrix H, (codewords) where elements of the code are all length $|L|$ strings z s.t. $Hz = 0$. (in binary)

Message length $= |L| - |R| = n - \frac{3}{4}n = \frac{1}{4}n$

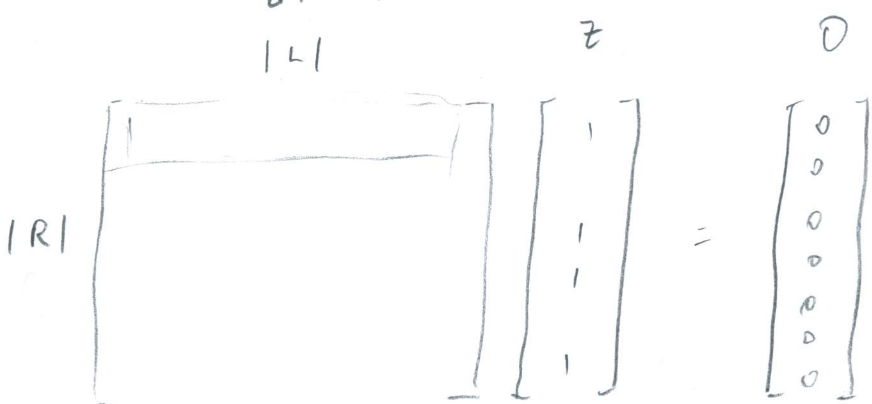
$[n, \frac{1}{4}n, ???]$ code : rate $= \frac{1}{4}$, constant

Claim Distance of the code is $> \frac{0.02}{64} n$

Assume minimum distance $\leq \frac{0.02}{64} n$

$\exists$ nonzero codeword z with Hamming weight $|z| \leq \frac{0.02}{64} n$. Let $S = \{ u \in [n] \mid z_u = 1 \}$ $\leftarrow$ vertices

Since z is nonzero, $S \neq \emptyset$, and hence $|S| \leq \frac{0.02}{64} n$.



Each vertex in R must be adjacent to an even number of vertices in S.

Claim $\cancel{0 \neq |S| \leq \frac{0.02}{64} n}$ There exists a $v \in N(S)$ with exactly one neighbor in S.

If all $v \in N(S)$ have ≥ 2 neighbors in S, then since $|N(S)| \geq 0.8 d |S|$:

$$|E(S, N(S))| \geq 2 |N(S)| \geq 2 \cdot 0.8 d |S| > 64 |S|$$

But the left partition is 64-regular!

Hence we are done. - z is not a codeword, so distance $> \frac{0.02}{64} n$.

Decoding is efficient! Flip bits of z that decrease the Hamming weight of Hz.

Corrects $< \frac{D}{2}$ errors in polylog n time

ERROR REDUCTION (Miller-Rabin primality)

Algorithm A, probabilistic
Uses n random bits, returns
fails with prob $p \leq 1\%$

YES if YES
NO 99%, YES 1% if NO

Old way of reducing error: repeat d times
• nd random bits, fails with prob $\leq .01^d$

Expanders:

Take the Margulis expander, except with $|L| = |R| = 2^n$.
Vertices on each sides are indexed by n-bit strings.

$$|N(S)| \geq 0.8 d |S| \quad \text{if } |S| \leq \frac{0.02}{d}(2^n)$$

Pick random vertex $v \in L$
Compute d neighbors of v and use them all for A!

Uses NO additional random bits, but what's the error...

How many 'bad' initial seeds are there?

Let $B_x \subseteq R$ be the set of 'bad' strings.

$|B_x| = p\, 2^n$

Let $C \subseteq L$ be the set of 'bad' choices $v \in L$ where $N(v) \subseteq B_x$

Claim: $|C| < \frac{0.02}{d} 2^n$

If not, take $S \subseteq C$ s.t. $|S| = \frac{0.02}{d} 2^n$

By expander properties, $|N(S)| \geq 0.8\, d\, |S|$

$= 0.8 \cdot 0.02 \cdot 2^n = 0.016 \cdot 2^n > |B_x|$ (because of p)

So error has been reduced to $\frac{0.02}{d}$ with <u>NO</u> additional random bits!

Random walks:

Start with random vertex v_1

Take a random walk $v_1\ v_2\ v_3\ v_4 \ldots v_m$.

Use the binary strings of v_i as a set

Due to the spectral properties of expanders, v_i are 'approximately' random.

Error $\approx 0.01^m$, random bits $n + m \log d$