18.434 Lecture Notes - LP/SDP Hierarchies II - Christopher Ford - April 22nd, 2016

Agenda

1) Review
   - general hierarchy scheme
   - Sherali-Adams (SA)
   - Lasserre
   - Lováz Schrijver

2) Applications to the knapsack problem

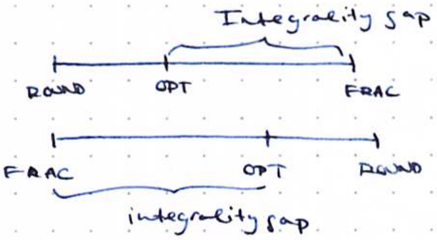
3) Questions / Misc.

Review

Big picture idea of heirarchies:
- working with optimization problems (MAX-CUT, MIN-VERTEX COVER, ETC.), often NP-hard
- true combinatorial optimum: OPT
- relax the problem to an LP/SDP and get: FRAC
- rounding procedure to get: ROUND

For maximization problems: ROUND $\leq$ OPT $\leq$ FRAC

For minimization problems: FRAC $\leq$ OPT $\leq$ ROUND



Int. gap = ratio of FRAC : OPT

- Big idea: add additional constraints so as to get better and better approximations

  often derived from the non-linear constraints of the original problem

  e.g. to force y_i be 0/1, want to write $y_i^2 = y_i \;\forall i$, but that isn't linear.
  write other constraints to approximate this requirement

- LP/SDP heirarchies define "levels / rounds" of a problem, each level higher more powerful than the ~~first~~ previous

  Note: for n variables $:\{0,1\}$, the n^{th} level has int. gap = 1 (i.e. exact solution)

- Lováz Schrijver
  - define an operator N on the convex relation P of a 0/1 linear program
  - each application of N adds additional variables and constraints
  - final relaxation: $N(N(\cdots N(P)\cdots)) := N^t(P)$ if applied t times
  - generated using the protection matrix of the decision variables
- Sherali Adams
  - fixes a problem in LS where the generation of successive protection matrices depends on ~~[illegible]~~ how/when additional variables were introduced
  - instead of introducing variables iteratively, they get added all at once
- Lasserre Heirarchy
  - provides a heirarchy for SDP relaxations of quadratic 0/1 programs
  - introduces a new variable for the product of every t variables in the original program

  e.g. Max ind. set would normally have $y_i \cdot y_j = 0 \;\forall (i,j) \in E$

  Now have y_S for each $S \subseteq V$, $y_S = 1$ iff all $i \in S$ are in ind. set

Before continuing; worth noting

1) in any of the added constraints / variables, integral solution still works. Thus, the process is indeed a relaxation

2) While the n^{th} level of the heirarchy provides the optimal integer solution, we've spent exponential time generating the additional constraints, so we may have just done the brute force $O(2^n)$ procedure.

Explicit linear formation for SA-Heirarchy

For any constraint $g_e(x) \geq 0$ in the original problem, we can write:

$$g_e(x) \prod_{i \in I} x_i \prod_{j \in J} (1-x_j) \geq 0$$ for any subsets $I, J \subseteq V$ where V is the variable set.

In plain english, we just multiply the constraint by some x_i and $1-x_j$ and sice each $x_i, x_j \in [0,1]$, $\prod_{i \in I} x_i \prod_{j \in J} (1-x_j) \in [0,1]$, so the above constraint will hold when $g_e(x) \geq 0$

If we duplicate a variable inside $g_e(x)$ in our products, then we'll get x_i^k in our expansion for $k \geq 1$ and i being the index of the variable duplicated.

Since $x_i \in \{0,1\}$ in our ILP, we can rewrite the integrality constraint as $x_i^k = x_i$ $\forall i, k$ and replace the x_i^k's in our expansion accordingly.

As Emanuele noted, we replace sie of the products with "big" variables. Namy, we replace each $\prod_{i \in S} x_i$ by Y_S, after having expanded.

Together, we can formally define the t-th level of the SA heirarchy as:
for any $1 \leq t \leq n$

$$SA^t(K) = \{ y \in P_t([n]) \mid y_\emptyset = 1 \text{ and } g'_{e,I,J}(y) \geq 0 \text{ for any } e \text{ and } I, J \text{ s.t. } |I \cup J| \leq t-1$$

where $g'_{e,I,J}(y)$ is obtained by: K: original polytope

1) mult $g_e(x)$ by $\prod_{i \in I} x_i \prod_{j \in J} (1-x_j)$
2) expand and replace x_i^k by x_i
3) replace each $\prod_{i \in S} x_i$ by y_S

A point $x \in [0,1]^V$ belongs to $SA^t(K)$ iff $\exists\, y \in SA^t(K)$ s.t. $y_{\{i\}} = x_i$ $\forall i \in V$

Knapsack Problem

given a set of n items, each w/ cost $c_i \geq 0$ and reward $r_i \geq 0$ and some knapsack capacity $C > 0$:
find a subset $S \subseteq [n]$ of cost $\sum_{i \in S} c_i \leq C$ that maximizes total reward $R = \sum_{i \in S} r_i$

ILP: $\max \sum_i r_i y_i$ (reward)

subject to
$\sum_i c_i y_i \leq C$ (capacity const)
$y_i \in \{0, 1\}$ (decision variable) integrality
$\forall i \in [n]$

$\xrightarrow{\text{LP relax}}$

$\max \sum_i r_i y_i$

subject to
$\sum_i c_i y_i \leq C$
$y_i \in [0, 1]$ (no longer integral)
$\forall i\ [n]$

Simple LP relaxation gives int. gap of 2: $\left(\frac{\text{FRAC}}{\text{OPT}}\right)$

Greedy solution performs badly: (i.e. sort by $\frac{r_i}{c_i}$, choose in order)
performs badly w/ 2 item case:

$c_1 = 1$, $r_1 = 2$ $\Rightarrow 2$ $\qquad$ $c_1 : n$, $r_1 : n$ $\Rightarrow 1$ $\qquad$ $n \gg 1$

$C = n$

picks the first object w/ no room for second. would have been better off picking only the second!

[Ibarra + Kim 1975]: Show that $\exists$ FPTAs that approximates Knapsack $(1+\epsilon)$
[Karlin, Mathieu, Nguyen, 2010]: show that even with good approximations, applying the SA heirarchy does not quickly reduce the integrality gap

Formally, the slow integrality convergence statement can be expressed as the following thm.

Thm. 1: for every $\epsilon, \delta > 0$, the integrality gap at the t'th level of the SA heirarchy for knapsack where $t \leq \delta n$ is at least $(2-\epsilon)(1/(1+\delta))$

Proof sketch: (full proof given as handout in lecture)

1) Consider an instance K of knapsack with n variables/objects
2) let K be an instance of knapsack where $c_i = r_i = 1 \quad \forall i$ (uniform)
3) let $C = 2(1-\epsilon)$ so we can only fit 1 object in the bag.

Thus, OPT = 1

4) Claim: the point y where $y_\emptyset = 1$ $\underbrace{y_{\{i\}} = C / (n + (t-1)(1-\epsilon))}_{\text{individual variables i.e. } x_1, x_2, \dots, x_n}$ $\underbrace{y_I = 0 \ \forall_I \text{ s.t. } |I| > 1}_{\text{products i.e. } x_1 x_2 x_3,\ x_3 x_4 x_{n-1} \dots}$

same definition as before i.e. Product ("big" variable)

is in $SA^t(K)$.

Note: this point gives $x_i = C / (n + (t-1)(1-\epsilon)) \quad \forall i \in [n]$

and thus $\sum_i r_i x_i = \sum_i 1 \cdot C / (n + (t-1)(1-\epsilon)) = Cn / (n + (t-1)(1-\epsilon))$

$$\text{gap} = \frac{Cn / (n + (t-1)(1-\epsilon))}{1} = Cn / (n + (t-1)(1-\epsilon)) \geq \frac{(2-\epsilon)}{(1+\delta)} \quad \text{when } t \leq \delta n$$

Highlevel proof of claim:

View the point y as a vector and then leverage the alt. definition of the t-th-SA lifted-polytope:

Nang: $SA^+(t)$ is the set of all vectors $y \in [0,1]^{P_t(V)}$ s.t. $y_\emptyset = 1$
$M_{P(U)}(y) \succeq 0$ and $M_{P(W)}(g_e \cdot y) \succeq 0$ $\forall$ e and subsets
$U, W \subseteq V$ s.t. $|U| \le t$, $|W| \le t-1$

$P_t(V)$: power set of V s.t. size is at most t.
$\succeq 0$ = PSD.

Pf. shows that the above definition holds for $y = y_\emptyset = 1$, $y_{\{i\}} = C/(n+(t-1)(1-\epsilon))$ $y_I = 0$ $\forall |I| > 1$

Key takeaway: applying SA to simple KS isn't as good as we thought.

What to do?

Rewrite LP as feasibility LP, then SA, gives much better int. gap.

Find $y_1, y_2, \dots, y_n$
such that

"lifted objective function"

$\sum_i y_i r_i \ge R$ $\quad g^1(y)$

$\sum_i y_i c_i \le C$ $\quad g^2(y)$

$0 \le y_i \le 1$ $\quad \forall i \in [n]$

Thm. 2

Integrality gap of t-level SA applied to the feasibility LP is $\le 1 + 1/(t-2)$

Key takeaway: by manipulating the LP before applying hierarchies, we can do much better.

sort $\frac{r_i}{c_i}$

Lemma 1: LP relax satisfies FRAC $\le$ GREEDY + $\max_i r_i$ (≤ 2OPT)

Let $S_{t-1} = \{ i \mid r_i > OPT/(t-1) \}$. at most $(t-2)$ items from S_{t-1} can fit in the bag. if $\exists i \in S_{t-1}$ w/ non-zero LP value, add constraint on LP to pick x_i.

at most, we repeat $(t-1)$ steps and the remaining items in S_{t-1} have LP value $= 0$.

K_0 = set of items chosen (non-zero LP value from before). $R_0 = r(K_0)$ reward.

Consider current LP solution Y' restricted to $[n] \setminus S_{t-1}$. New LP value $=$ $\sum_{i \notin S_{t-1}} Y'_i \ge R - R_0$. apply greedy alg. to remaining items, gives additional reward R_g. from Lemma 1:

$$R - R_0 \le \sum_{i \notin S_{t-1}} Y'_i \le R_g + \max_{i \notin S_{t-1}} r_i < R_g + OPT/t-1 \le R/(t-1)$$

Value of rounded solution $R_0 + R_g \ge R\left(1 - \frac{1}{t-1}\right)$