Linear Programming

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1) Formulation:

$$\underbrace{\begin{array}{l} \max f(x) \\ \text{s.t. } x \in X \end{array}}_{\text{General optimization}} \qquad \underbrace{\begin{array}{l} \max f(x) \\ \text{s.t. } g_i(x) \leq 0 \end{array}}_{\text{General optimization w/ constraints}} \qquad \underbrace{\begin{array}{l} \max c^T x \\ \text{s.t. } Ax \leq b \\ \quad\ x_i \geq 0 \end{array}}_{\text{Linear program (standard form)}}$$

$\max c^T x \longrightarrow$ objective function

s.t. $Ax \leq b \longrightarrow$ feasible set/region $\{x : Ax \leq b,\ x_i \geq 0\}$.

$x_i \geq 0$

* An LP need not look like the standard form. For example:

$$\begin{array}{l} \min c^T x \\ \text{s.t. } Ax \leq b \\ \quad\ x_i \geq 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{l} \max (-c)^T x \\ \text{s.t. } Ax \leq b \\ \quad\ x_i \geq 0 \end{array}$$

is a linear program

$$\begin{array}{l} \max c^T x \\ \text{s.t. } Ax = b \\ \quad\ x_i \geq 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{l} \max c^T x \\ \text{s.t. } \begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix} \\ \quad\ x_i \geq 0 \end{array}$$

is also a linear program.

* A general class of problems can be considered LP.

2) Applications: LP is a natural formulation for many problems:

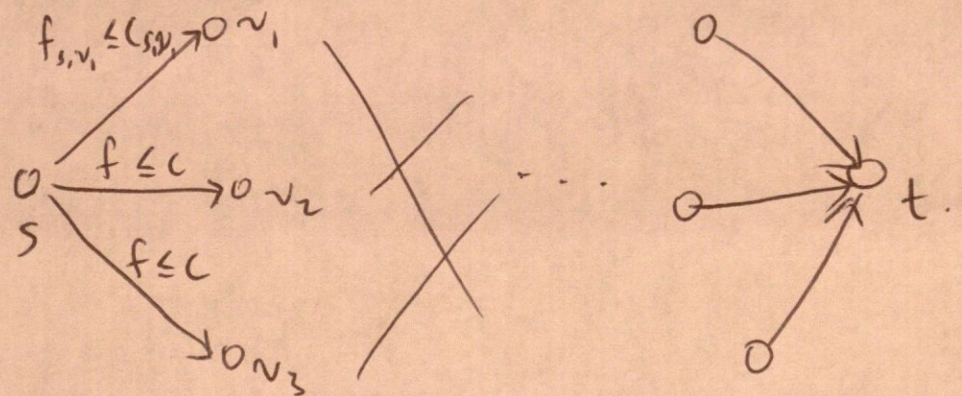
Ex 1: Mixed resources

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 15 & 0 \\ 5 & -5 & -3 \end{pmatrix} \begin{pmatrix} \text{classes} \\ \text{friends} \\ \text{sleep} \end{pmatrix} \leq \begin{pmatrix} \text{time} \\ \text{money} \\ \text{sanity} \end{pmatrix} = \begin{pmatrix} 24 \\ 100 \\ 20 \end{pmatrix}, \quad x_i \geq 0$$

$$\text{maximise happiness} = c_1 \times \text{classes} + c_2 \times \text{friends} + c_3 \times \text{sleep}$$

Ex 2: Max flow

Let $G=(V,E)$ be a directed graph, each edge $(u,v) \in E$ allows a maximum "flow" across it, given by a capacity constraint. Want to maximise total flow from $s \in V$ to $t \in V$.

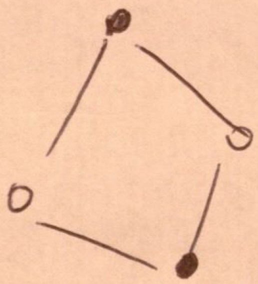


$$\max \sum_{v:(s,v)\in E} f(s,v) \longrightarrow \text{maximize flow}$$

$$\text{s.t.} \sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w) \quad \forall v \in V \setminus \{s,t\} \longrightarrow \text{flow in} = \text{flow out}$$

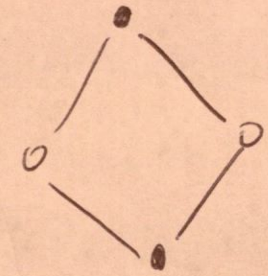
$$0 \leq f(u,v) \leq c_{u,v} \longrightarrow \text{max/min capacity constraints.}$$

Ex 3: Vertex cover

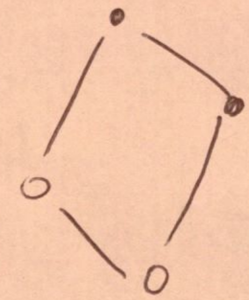


• $\in S$

∘ $\notin S$



Is a vertex cover



Is <u>NOT</u> a vertex cover

$G = (V, E)$

$S \subseteq V$

Want to find $\min |S|$ such that every edge is adjacent to at least one vertex in S.

$$\min \sum_{v \in V} x_v \qquad \overset{\text{relaxation}}{\rightsquigarrow} \qquad \min \sum_{v \in V} x_v$$

$$x_u + x_v \geq 1 \quad \forall (u,v) \in E \qquad\qquad x_u + x_v \geq 1$$

$$x_v \in \{0,1\} \quad v \in V \qquad\qquad 0 \leq x_v \leq 1.$$

ILP is HARD

$\in$ NP-hard

Pros: solving this gives exact solution

Cons: solving this is hard

LP is EASY

$\in$ P

Pros: solving this is easy

Cons: May get bad solutions

4) Geometry:

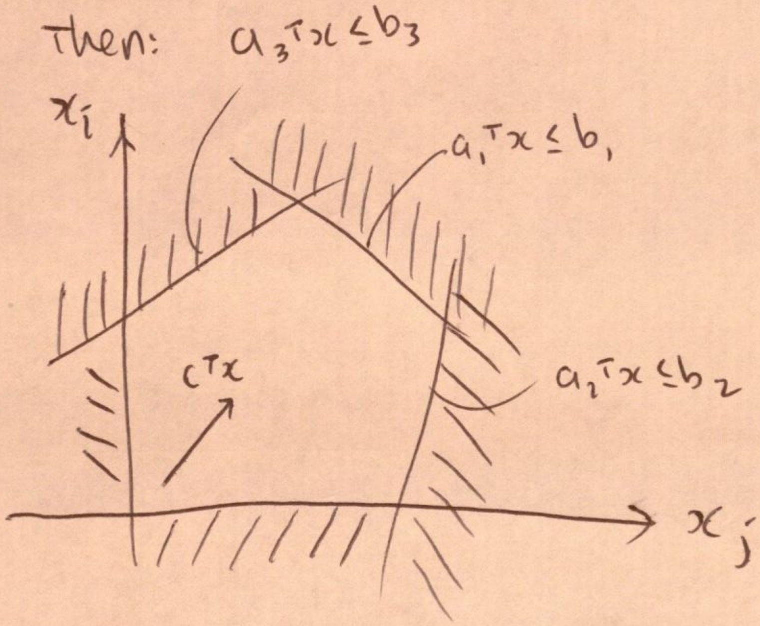
Consider $\{ x : Ax \leq b \} \Leftrightarrow \{ x : a_i^T x \leq b_i \ , i \in \{1 \ldots m\} \}$

Each $a_i^T x = b$ denotes a hyperplane in $\mathbb{R}^n$

Each $a_i^T x \leq b$ denotes a half-space in $\mathbb{R}^n$

$c^T x$ is a vector in $\mathbb{R}^n$

Then:



Feasible region forms a <u>polytope</u> in $\mathbb{R}^n$.

Solving LPs involve travelling along the edges of the polytope (simplex method), or travelling in the interior of the polytope (interior point methods).

5) Duality

(Primal)	(Dual)
$\max c^T x$	$\min b^T y$
s.t. $Ax \leq b$	$A^T y \leq c$
$x \geq 0$	$y \geq 0$

(?) How do we get the dual?

(A) "multiply equations by constants"

$\max 2x_1 + 3x_2 + 4x_3$

$$\begin{array}{c} y_1 \times \\ y_2 \times \\ y_3 \times \end{array} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 10 \\ 9 \\ 8 \end{pmatrix}, \quad x_i \geq 0$$

$$y_1 (x_1 + x_2 + x_3) \leq 10 y_1$$
$$y_2 (0 - 5x_2 + x_3) \leq 9 y_2$$
$$y_3 (x_1 + x_2 + 2x_3) \leq 8 y_3$$

$\Rightarrow$ $10y_1 + 9y_2 + 8y_3$ provides an upper bound on the objective if

$$y_1 + y_3 \geq 2$$
$$y_1 - 5y_2 + y_3 \geq 3$$
$$y_1 + y_2 + 2y_3 \geq 4$$

$\rightsquigarrow$ $\min b^T y$, $A^T y \leq c$, $y \geq 0$

Theorem: weak duality:

$$c^T x \le b^T y.$$

Proof: $c^T x \le y^T A x \le y^T b = b^T y$ (qed).

Theorem: strong duality for problems w/ Linear constraints:

let the objective function f be convex over $\mathbb{R}^n$ and let the constraint set be polyhedral. If the optimal value f^* is finite, then there is no duality gap.

Proof: use lagrangian multipliers. If the primal is:

$$(P) \quad \min f(x)$$
$$\text{s.t. } g_i(x) \le 0$$
$$x_i \geq 0$$

let $\mathcal{L}(x, \lambda) = f(x) + \sum_i \lambda_i g_i(x)$

Define the dual function as:

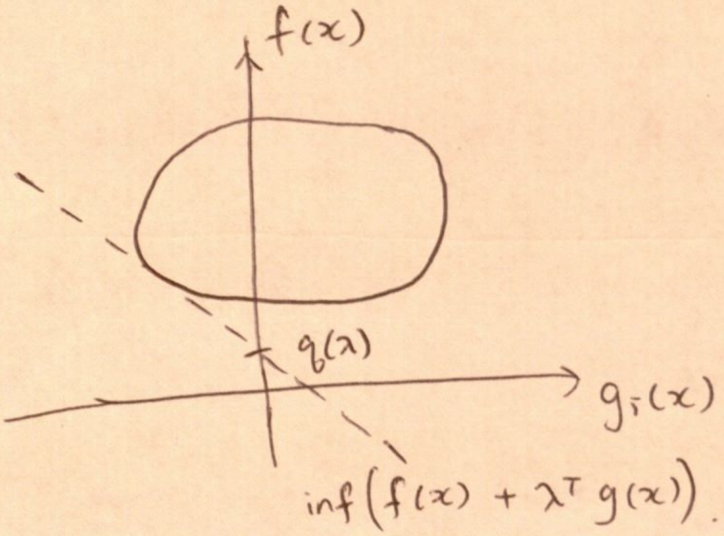
$$q(\lambda) = \inf_{x \in \mathbb{R}^n_+} \mathcal{L}(x, \lambda)$$

Then the dual problem is:

$$(D) \quad \max q(\lambda)$$
$$\text{s.t. } \lambda \geq 0.$$

Proof (cont.):

we can draw the dual function as a supporting hyperplane

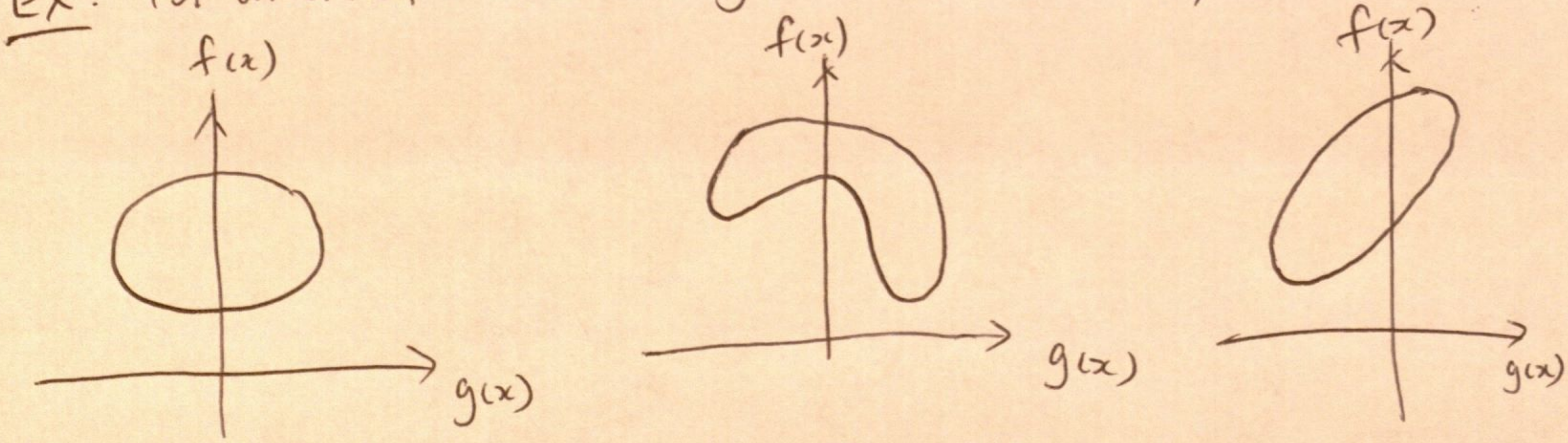


~~Sing~~ Since $g_i(x) \le 0$, we only care about the left part of the feasible region. Now, if we vary the slope of ~~of~~ the supporting hyperplane by varying λ, it is clear that

$$\max_{\lambda} q(\lambda) = \min_{x} f(x)$$

when the feasible region is convex. (qed).

Ex: For which of these diaglams is the duality gap 0?



strong duality? Y N. Y

Applications of duality:

Take the dual of the max-flow ~~pr~~ formulation:

$$\min \sum_{(u,v)\in E} c_{u,v}\, y_{u,v}$$

$$\text{s.t.} \sum_{(u,v)\in P} y_{u,v} \geq 1 \quad \forall_P \in P.$$

$$y_{u,v} \geq 0$$

$\Rightarrow$ is the min-cut formulation!

Theorem: Max flow - min cut

In a graph, max flow = min cut

Proof: The proof is obvious by strong duality, since the duality gap is 0. (qed).