LP Relaxations & Approximation Algorithms.

Vertex Cover. Input: undirected graph $G=(V,E)$

Goal: Find $S \subseteq V$ so that 1) $\forall \{i,j\} \in E$, $\{i,j\} \cap S \neq \emptyset$, 2) $|S|$ minimized

NP-Hard to find the minimum VC of a graph.

Goal Approximation — find a solution that is "comparable" with the minimum VC.

α-approximation. An algorithm is α-approximation for VC (or any other minimization problem) if it always outputs a solution with objective value $ALG \leq \alpha \cdot OPT$ $(\alpha \geq 1)$.

Integer Linear Program for Vertex Cover. Let $x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$

ILP:
$$\begin{aligned} \text{minimize} \quad & \sum_{i \in V} x_i \\ \text{s.t.} \quad & x_i \in \{0,1\} \quad \forall i \in V \\ & x_i + x_j \geq 1 \quad \forall \{i,j\} \in E \end{aligned}$$

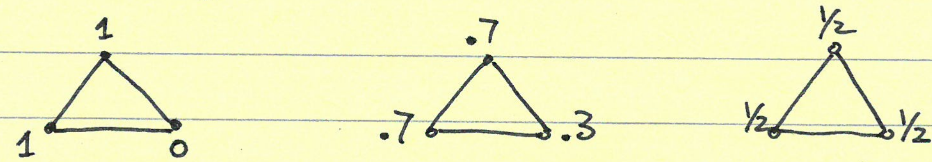
Still NP-Hard to solve the ILP — because of the integral constraint $x_i \in \{0,1\}$

Relaxation: $x_i \in \{0,1\} \xrightarrow[\text{relax}]{} x_i \in [0,1]$ (a weaker constraint).

LP relaxation for Vertex Cover.

LP:
$$\begin{aligned} \text{minimize} \quad & \sum_{i \in V} x_i \\ \text{s.t.} \quad & 0 \leq x_i \leq 1 \quad \forall i \in V \\ & x_i + x_j \geq 1 \quad \forall \{i,j\} \in E \end{aligned}$$

- Efficient (poly-time) solvable!
- Every solution for ILP remains valid in LP (because of relaxation): $LP \leq ILP$
- However there might be new fractional solutions.



integral solution for ILP

new fractional solutions.

: optimal solution for $LP = 1.5$ what does it mean in Vertex Cover?

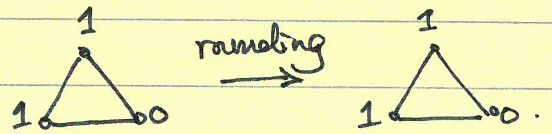
Rounding procedure. Take a fractional solution, and convert it to a integral solution. (But may lose some value in objective function).

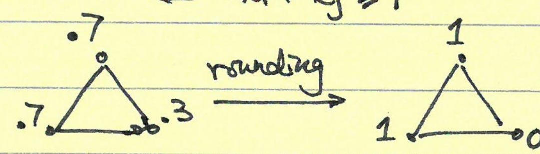
Rounding for Vertex Cover. Let $x_i^* = \begin{cases} 1 & \text{if } x_i \geq 1/2 \\ 0 & \text{if } x_i < 1/2 \end{cases}$

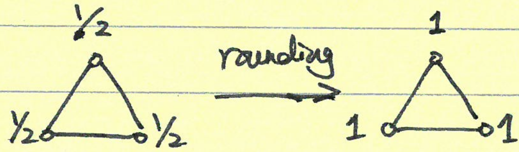
Claim $\{x_i^*\}$ is valid soln. for ILP.

Proof. 1) $x_i^* \in \{0,1\}$ $\forall i \in V$; 2) $x_i^* + x_j^* \geq 1$ $\forall \{i,j\} \in E$ $\Leftarrow$ at least one of $x_i, x_j \geq 1/2$

$\Leftarrow x_i + x_j \geq 1$







What about objective value?

Claim ~~Rounding ≤ 2.~~ $\sum_{i \in S} x_i^* \leq 2 \cdot \sum_{i \in S} x_i$

Proof. Because $x_i^* \leq 2 \cdot x_i$ $\forall i \in \cancel{S}V$.

Claim. Solving LP relaxation + Rounding is 2-approx for Vertex Cover.

Proof Rounding $\leq 2 \cdot LP \leq 2 \cdot ILP$

Integrality Gaps. We proved that $LP \in [\frac{1}{2} ILP, ILP]$ (to see the lower bound: $ILP \leq$ rounding $\leq 2 \cdot LP$).

- LP is poly-time solvable, ILP is NP-Hard. LP is an "estimator" of ILP.
- How good is this estimator? — Up to a factor of 2.
- Can we improve the analysis to make it better? — Can't be better than 4/3. (the triangle graph)
- Consider K_n — complete graph with n vertices.

  ILP(min VC) $= n-1$ $LP \leq n/2$ ($x_i = \frac{1}{2}$ $\forall i$ is a feasible soln.)

  $ILP/LP \geq 2 - O(\frac{1}{n})$. — an example where LP is indeed off by factor of 2

~~Integrality gap instance is a certificate~~

We say K_n is a 2-integrality gap instance for LP. — a certificate on the bad estimation of LP.

Set Cover Input: Universe $\mathcal{U} = \{1, 2, \cdots, n\}$ and $S_1, S_2, \cdots, S_m \subseteq \mathcal{U}$

Goal: Find smallest $I \subseteq \{1, 2, \cdots, m\}$, s.t. $\bigcup_{i \in I} S_i = \mathcal{U}$

Remark Vertex Cover is a special case of Set Cover. (Universe is the set of edges, each set corresponds to the set of incident edges of a vertex)

ILP for set cover: Let $x_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{o/w} \end{cases}$

$$\text{Minimize} \quad \sum_{i=1}^{m} x_i$$

$$\text{s.t.} \quad x_i \in \{0, 1\}, \quad \forall i \in \{1, 2, \cdots, m\}$$

$$\sum_{i: u \in S_i} x_i \geq 1 \quad \forall u \in \mathcal{U}$$

LP relaxation.

$$\text{minimize} \quad \sum_{i=1}^{m} x_i$$

$$\text{s.t.} \quad x_i \in [0, 1] \quad \forall i \in [m]$$

$$\sum_{i: u \in S_i} x_i \geq 1 \quad \forall u \in \mathcal{U}.$$

Claim ILP $\geq$ LP (because of relaxation).

Rounding. Idea: given LP solution $\{x_i\}$, treat x_i as the probability that $i \in I$

Alg RandomPick - For each $i \in [m]$, let $i \in I$ w.p. x_i independently.
- Return I.

Claim. $\mathbb{E}|I| = \sum_{i=1}^{m} x_i$

Claim For each $u \in \mathcal{U}$, $\Pr[u \in \bigcup_{i \in I} S_i] \geq 1 - 1/e$.

Proof. $\Pr[u \notin \bigcup_{i \in I} S_i] = \prod_{i: u \in S_i} \Pr[i \notin I] = \prod_{i: u \in S_i} (1 - x_i)$

$\leq \prod_{i: u \in S_i} e^{-x_i}$ (we use the fact $1 - t \leq e^{-t}$ $\forall t \geq 0$)

$= \exp\left(-\sum_{i \in I: u \in S_i} x_i\right)$

$\leq \exp(-1)$ (because of the LP constraint $\sum_{i: u \in S_i} x_i \geq 1$)

Remark. RandomPick returns a set I that covers each element w.p. $\geq 1 - 1/e$.
Repeat a few times so that each element covered with higher prob.

Alg RandomizedRound. - Iterate $\lceil 2 \ln n \rceil$ times, at iteration j, let $I_j \leftarrow$ RandomPick
- Return $I = \bigcup_{j=1}^{\lceil 2 \ln n \rceil} I_j$.

Claim $\mathbb{E}|I| \leq \sum_j \mathbb{E}|I_j| \leq \lceil 2\ln n \rceil \cdot \sum_{i=1}^{m} x_i \leq \lceil 2\ln n \rceil \cdot OPT$.

Claim For each $u \in \mathcal{U}$, $\Pr[u \in \bigcup_{i\in I} S_i] \geq 1 - \prod_{j=1}^{\lceil 2\ln n \rceil} \Pr[u \notin \bigcup_{i \in I_j} S_i]$

$$\geq 1 - (e^{-1})^{\lceil 2\ln n \rceil} \geq 1 - 1/n^2$$

Corollary $\Pr[I \text{ covers } \mathcal{U}] = 1 - \Pr[\exists u : u \notin \bigcup_{i\in I} S_i]$

$$\geq 1 - \sum_{u\in\mathcal{U}} \Pr[u \notin \bigcup_{i\in I} S_i] \qquad \text{(union bound)}$$

$$\geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}.$$

Theorem. W.p. $1/2 - 1/n \geq .4$ (for large enough n), RandomizedRound returns I s.t. $|I| \leq OPT \cdot (4\ln n + O(1))$, & I is a set cover for $\mathcal{U}$.

Proof. 1) $\Pr[|I| \leq OPT \cdot (4\ln n + \cancel{O(1)}\ 2)] = 1 - \Pr[|I| > (4\ln n + 2) \cdot OPT]$

$$\geq 1 - \frac{\mathbb{E}|I|}{(4\ln n + 2)\cdot OPT} \quad \text{(Markov Ineq.)}$$

$$\geq 1 - 1/2 = 1/2$$

2) $\Pr[I \text{ covers } \mathcal{U}] \geq 1 - \frac{1}{n}$ by the corollary.

Therefore, $\Pr[|I| \leq OPT(4\ln n + 2) \ \& \ I \text{ covers } \mathcal{U}]$

$$\geq 1 - (1 - 1/2) - (1 - (1 - 1/n)) = 1/2 - 1/n \qquad \text{(union bound)}.$$