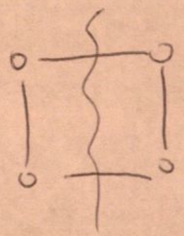
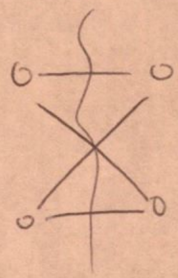
Max Cut

$G=(V,E)$, find $S \subseteq V$ such that the cardinality of the cut-set is maximized. i.e. $C=\{(u,v)\in E : u\in S,\ v\in V\setminus S\}$, $\max |C|$.



not max,



max

max.

Thm: (the decision version of) max-cut is NP-complete [Karp 1972].

ILP formulation: Let's have a variable x_{uv} for each edge (u,v) and a variable y_u for each vertex $u \in V$.

ILP

$$\max \sum_{(u,v)\in E} x_{uv}$$

s.t. $x_{uv} \le y_u + y_v \quad \forall (u,v)\in E$

$x_{uv} \le 2-(y_u+y_v)$

$x, y \in \{0,1\}$.

LP

$\xrightarrow{\text{relaxation}}$ $0 \le x, y \le 1$

If $y_u = y_v$, then $x_{uv}=0$. If $y_u \ne y_v$, then $x_{uv}=1$.

So x_{uv} is an indicator variable for edges in the cut-set for the ILP formulation

A few observations:

① Since this is a maximization problem, so $LP \geq ILP = OPT$

② If $y_v = \frac{1}{2}$ $\forall v \in V$, then $x_{uv} = 1$ $\forall (u,v) \in E$. So this is the LP optimal solution! So $LP = |E|$

③ With a naive, greedy algorithm, we have max-cut $\geq \frac{1}{2}|E|$. So $ILP \geq \frac{1}{2}|E|$.

Putting these together, we have: $\underbrace{OPT \leq LP \leq 2\,OPT}_{\text{Integrality gap.}}$

Rounding of LP solution

Randomized rounding scheme: for each y_v, put $v \in S$ with probability y_v.

Then the expected # edges cut is:

$$\sum_{(u,v)\in E} \Pr[(u,v) \text{ in cut}] = \sum_{(u,v)\in E} y_u(1-y_v) + y_v(1-y_u).$$

Now, for all $y_u, y_v \in [0,1]$, we have

$$y_u(1-y_v) + y_v(1-y_u) \geq \frac{1}{2}\min\{y_u + y_v,\ (1-y_u)+(1-y_v)\}$$
$$= \frac{1}{2}x_{uv}$$

so we have $\mathbb{E}(\text{rounding}) \geq \frac{1}{2}LP \geq \frac{1}{2}OPT$

$\Rightarrow$ ½-approximation for max-cut!

Turns out all LP relaxations + rounding schemes are ½-approximations at best, so LP is pretty bad. ~~Need to try something else~~.

Also the optimal solution for the LP relaxation is trivial, so that's pretty stupid. So we need to try something else.

Quadratic Unconstrained Binary Optimization (QUBO).

We present a different formulation of the max-cut problem:

QUBO

$$\max \frac{1}{2} \sum_{(u,v)\in E} (1 - x_u x_v) \rightsquigarrow \min \; x^T Q x$$

$\uparrow$ adjacency matrix of G.

s.t. $x_u \in \{-1, 1\} \quad \forall u \in V.$

We want to find an SDP relaxation for QUBO.

#1 Rank relaxation

$$x^T Q x = \mathrm{Tr}(x^T Q x) = \mathrm{Tr}(Q x x^T) = \mathrm{Tr}(QX).$$

For the QUBO problem, X has properties:

$$X \succeq 0 \qquad X_{ii} = x_i^2 = 1 \qquad \text{and} \qquad \mathrm{rank}(X) = 1.$$

so

$$\begin{aligned} \min \;& \mathrm{Tr}\, QX \\ \text{s.t. } & X_{ii} = 1 \\ & X \succeq 0 \\ & \mathrm{rank}(X) = 1 \end{aligned} \xrightarrow{\text{rank relaxation}} \boxed{\begin{aligned} \min \;& \mathrm{Tr}\, QX \\ \text{s.t. } & X_{ii} = 1 \\ & X \succeq 0 \end{aligned}}$$

SDP formulation

* Allowing $\mathrm{rank}(X) \geq 1$ is the relaxation step, that produces the SDP formulation. This is known as "lifting", as though into a higher dimension.

#2

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#2 Lagrangian duality

$$\boxed{\begin{aligned} &\min\ x^T Q x \\ &\text{s.t. } x_i^2 - 1 = 0 \end{aligned}}$$ original QUBO form

we find the lagrangian:

$$\mathcal{L}(x,\lambda) = x^T Q x - \sum_{i=1}^{n} \lambda_i (x_i^2 - 1)$$

$$= x^T (Q - \Lambda) x + \operatorname{Tr} \Lambda$$

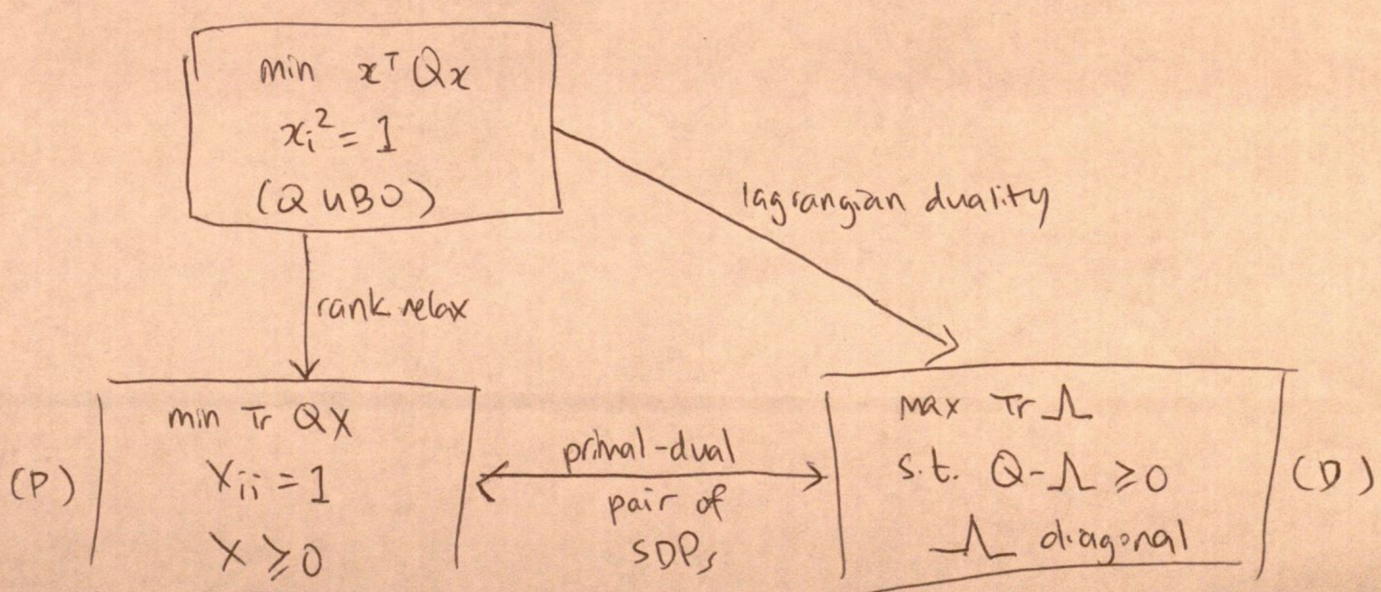
Then the dual function is $q(\lambda) = \inf_x \mathcal{L}(x,\lambda)$. For this to be bounded below, we need the implicit constraint that $Q - \Lambda \succeq 0$

Then $\inf_x \mathcal{L}(x,\lambda) = \operatorname{Tr}\Lambda$, so the dual problem is

$$\boxed{\begin{aligned} \max\ q(\lambda) = \max\ &\operatorname{Tr}\Lambda \\ \text{s.t. } &Q - \Lambda \succeq 0 \\ &\Lambda \text{ is diagonal} \end{aligned}}$$

This is an SDP! We added the constraint that $Q - \Lambda \succeq 0$.

Note that if we take the dual of this SDP, we obtain the primal version in (#1 Rank relaxation)



Goemans - Williamson Rounding

Now we have the SDP formulation, for which we can find the optimum solution X. How do we recover the original assignment for max cut? Goemans and williamson tell us a way to do this.

① Factorize $X = V^T V$, $V \in \mathbb{R}^{r \times n}$, not necessarily rank 1!

$$V = \begin{bmatrix} v_1 \cdots & v_i \cdots & v_n \end{bmatrix} \updownarrow r \quad , \text{each } X_{ij} = v_i^T v_j$$

② Assign each v_i to a point on the unit sphere in $\mathbb{R}^r$

③ Choose a random hyperplane, and assign each x_i^* to be $+1$ or -1 depending on which side of the hyperplane v_i lies on. (Fig 1).

What is the expected value of this rounding scheme? We have

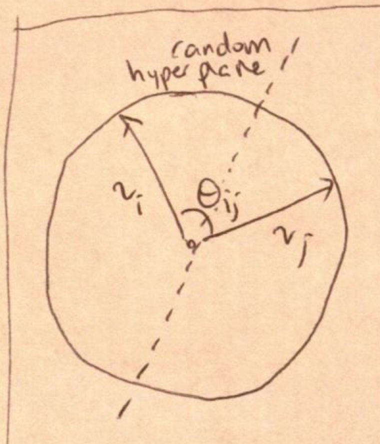


Fig 1, unit sphere in $\mathbb{R}^r$

$$\begin{aligned} ALG &= \mathbb{E}\left(\tfrac{1}{2} \textstyle\sum (1 - x_i^* x_j^*)\right) \\ &= \tfrac{1}{2} \textstyle\sum \mathbb{E}(1 - x_i^* x_j^*) \\ &= \tfrac{1}{2} \textstyle\sum [2 \times \Pr(v_i, v_j \text{ are on different sides}]) \\ &= \tfrac{1}{2} \textstyle\sum \tfrac{2}{\pi} \theta_{ij} \end{aligned}$$

Also, $SDP = \frac{1}{2} \sum (1 - X_{ij})$. We want ~~to~~ to find α such that: $\alpha SDP \leq ALG \leq SDP$.

Consider that:

$$\begin{aligned} \alpha \left(\tfrac{1}{2} \textstyle\sum (1 - X_{ij})\right) &\leq \tfrac{1}{2} \cdot \tfrac{2}{\pi} \textstyle\sum \theta_{ij} \\ \alpha (1 - X_{ij}) &\leq \tfrac{2}{\pi} \theta_{ij} \\ &= \tfrac{2}{\pi} \arccos(v_i^T v_j) \\ &= \tfrac{2}{\pi} \arccos(X_{ij}) \end{aligned}$$

$\alpha = 0.878$ will satisfy this inequality for all $X_{ij} \in [0, 1]$!

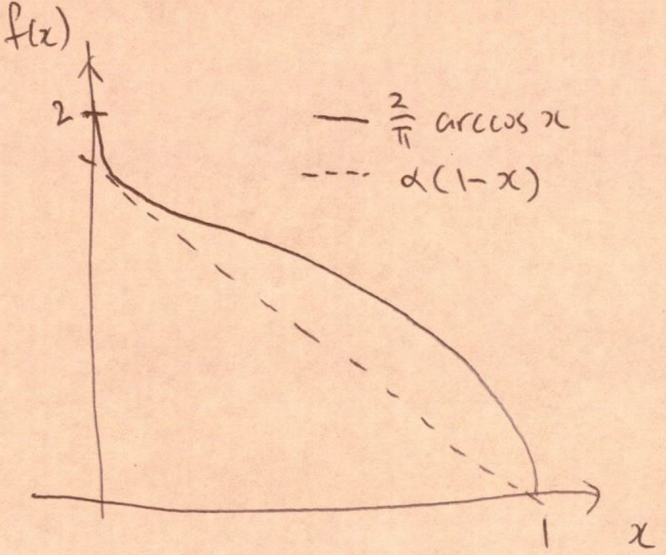


Fig 2: Find α.

So for $\alpha = 0.878$ we have

$$\alpha SDP \le ALG \le SDP.$$

Finally, to find the bounds on OPT we ~~ha~~ observe that:

$OPT \le SDP$ (relaxation)

$OPT \geq ALG$ (since OPT is max).

Putting together these inequalities we have:

$$\alpha SDP \le ALG \le OPT \le SDP$$

$$\boxed{\alpha OPT \le ALG \le OPT}$$

So we have found a 0.878-approximation for max-cut! ∎.

Other results

Def. An algorithm is an (a, b)-approximation if, if $OPT = a$, then $ALG = b$.

Ex: Goemans Williamson is a $(c, 0.878c)$-approximation for max-cut, as we have proved above.

Now if we normalize max-cut (such that $OPT = 1$ means all edges are cut), then we can prove that GW is also a $(1-\varepsilon, 1-\Omega(\sqrt{\varepsilon}))$-approximation.

Proof: Given SDP solution $X = V^T V$, we have our SDP optimum at:

$$SDP = \sum_{(i,j)\in E} \frac{1 - v_i^T v_j}{2|E|} \geq OPT = 1 - \varepsilon \quad , \quad \varepsilon > 0$$

Note that this SDP is normalized with factor $\frac{1}{|E|}$. Now we can define ε_{ij}:

$$\varepsilon_{ij}: \; 1 - \varepsilon_{ij} = \frac{1 - v_i^T v_j}{2}$$

Then we have $SDP = \sum_{(i,j)\in E} \frac{1-\varepsilon_{ij}}{|E|} = 1 - \sum_{(i,j)\in E} \frac{\varepsilon_{ij}}{|E|} \geq 1 - \varepsilon.$

Now we can follow the proof for GW rounding to find $\mathbb{E}$(rounding):

$$\mathbb{E}(\text{rounding}) = ALG = \frac{1}{|E|} \sum_{(i,j)\in E} \Pr[\, v_i^T, v_j \text{ are on different sides}]$$

$$= \frac{1}{|E|} \sum_{(i,j)\in E} \frac{\arccos(v_i^T v_j)}{\pi}$$

$$= \frac{1}{|E|} \sum_{(i,j)\in E} \frac{\arccos(2\varepsilon_{ij} - 1)}{\pi}$$

Using a lemma from Taylor, we see that:

$$\boxed{\begin{aligned} \cos(\pi - \delta) &\approx -1 + \delta^2 \\ \frac{\arccos(\delta^2 - 1)}{\pi} &\approx 1 - \frac{\delta}{\pi} \geq 1 - \Omega(\delta) \end{aligned}}$$

So
$$ALG \geq \frac{1}{|E|} \sum_{(i,j)\in E} 1 - \Omega(\sqrt{\varepsilon_{ij}})$$

$$= 1 - \frac{1}{|E|} \sum_{(i,j)\in E} \Omega(\sqrt{\varepsilon_{ij}})$$

and by the minkowski inequality (?), this is upper bounded by

$$\geq 1 - \Omega\left(\sqrt{\sum_{(i,j)\in E} \frac{\varepsilon_{ij}}{|E|}}\right).$$

$$\geq 1 - \Omega(\sqrt{\varepsilon}).$$

$\square$.