Spectral Graph Theory I - Christopher Ford 2/10/2015

Agenda: 1) Graph theory review and vertex functions

2) Local variance

3) Random walks + random vertices

4) Inner products (time permiting)

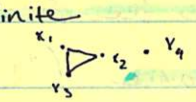
Graph Theory Review

- A graph G is defined by its vertices and edges.

  $G = (V, E)$ where V is the set of vertices, E the set of edges

- for our graphs we assume the following:

  1) V, E are finite $\Rightarrow$ G is finite

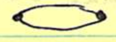
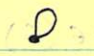
  2) no isolated vertices e.g. 

     (no vertices of degree zero)

  3) the edges do not have weights and are undirected

     (though we can augment our analysis w/ parallel edges to account for this)

NOTE: parralel edges and self-loops are allowed

self-loops can be thought of as ½ edges and contibute 1 to the degree of the vertex they are connected to.

NOTE: degree = # edges adjacent to a vertex. if degrees of all vertices are the same, the graph is regular

Vertex Functions

- useful to label vertices w/ a function

  $f: V \to \mathbb{R}$ e.g. 6.046 cows probleem, voltages, indicator

  indicator function: 0/1 if $v_i \in S$, $S \subseteq V$

$$f: V \to \mathbb{R} \equiv \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix}$$ can be thought of as a vector

Vector Functions Continued

note: addition and scaling of these functions are preserved:

$$(f+g)(v_i) = f(v_i) + g(v_i)$$

$$(c \cdot f)(v_i) = c \cdot (f(v_i))$$

$\Rightarrow$ linearity of functions

Local Variance

** the main idea of spectral graph theory **

In simple terms:

We have a function f that maps $V \to \mathbb{R}$. Want to know how much the function varies between adjacent vertices in the graph.

Def: local variance ("Dirichlet form", "analytic boundary size")

$$\mathcal{E}(f) = \frac{1}{2} \underset{u \sim v}{E}\left[(f(u) - f(v))^2\right]$$

$u \sim v$ probability dist. across edges

Immediate observations:

$$\left.\begin{array}{l} 1)\ \mathcal{E}(f) \geq 0 \\ 2)\ \mathcal{E}(c \cdot f) = c^2 \mathcal{E}(f) \\ 3)\ \mathcal{E}(f+c) = \mathcal{E}(f) \end{array}\right\}$$ same properties as original/traditional variance

#1 is trivial. #2: $\mathcal{E}(c \cdot f) = \frac{1}{2} \underset{u \sim v}{E}\left[(c \cdot f(u) - c \cdot f(v))^2\right]$

$$= \frac{1}{2} \underset{u \sim v}{E}\left[(c(f(u) - f(v)))^2\right]$$

$$= \frac{c^2}{2} \underset{u \sim v}{E}\left[(f(u) - f(v))^2\right] = c^2 \mathcal{E}(f)$$

#3: $\mathcal{E}(c+f) = \frac{1}{2} \underset{u \sim v}{E}\left[(f(u) + c - f(v) - c)^2\right]$

$$= \frac{1}{2} \underset{u \sim v}{E}\left[(f(u) - f(v))^2\right] = \mathcal{E}(f)$$

intuition for local variance:

- $\mathcal{E}(f)$ is small when f doesn't differ much b/t adjacent vertices. f is "smooth" across ~~...~~ edges
- $\mathcal{E}(f)$ is large in the other case. f is "anti-smooth" or "rough"

(not formal terms)

Example (time permiting)

$$\underbrace{f(v) = \begin{cases} 1 & v \in S \\ 0 & v \notin S \end{cases}}_{\text{indicator function}} \qquad S \subseteq V$$

$f(v) = 1_S$

$$\mathcal{E}(f) = \frac{1}{2} \mathop{E}_{u \sim v} \left[(f(u) - f(v))^2 \right]$$

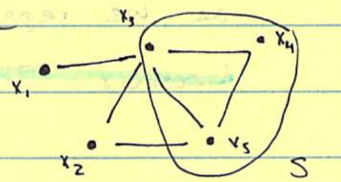
$$= \frac{1}{2} \mathop{E}_{u \sim v} \left[(1_S(u) - 1_S(v))^2 \right]$$

NOTE $(1_S(u) - 1_S(v))^2 = 1$ iff one of the vertices $\in S$, but not the other. 0 otherwise

$$= \frac{1}{2} \mathop{E}_{u \sim v} \left[1((u,v) \text{ "crosses" } S) \right]$$

↳ enters / exits the subset

$$= \Pr[u \to v \text{ "steps" out of } S]$$



$x_3, x_4, x_5 \in S$

$x_1 \sim x_3$, $x_2 \sim x_3$, $x_2 \sim x_5$ "cross" S

Random Vertices + Random Walks

π : distribution of randomly selected vertices chosen by the following procedure

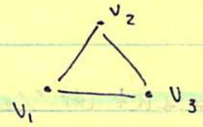
1) choose a random edge $u \sim v$
2) output u (or v, identical by symmetry)

think of π as a distribution across vertices weighted by their degree

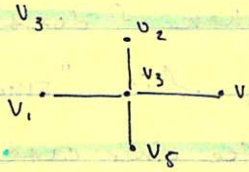
more degree ⟶ more likely to be chosen in step 1.

$$\pi[v] = \frac{\deg(u)}{|E|} \times \frac{1}{2}$$ (pick edge adj. to u, output u as endpoint)

example:



$\pi(v_1) = \pi[v_2] = \pi[v_3] = \frac{1}{3}$



$\pi(v_1) = \pi[v_2] = \pi[v_4] = \pi[v_5] = 1/8$

$\pi[v_3] = \frac{4}{2 \cdot 4} = 1/2$

Application to random walks:

① picking u from π then picking v at a uniformly random neighbor of u is the same as ② drawing an edge uniformly at random $u \sim v$

Pf.

$$\text{prob of getting } uv \text{ from ①} = \underbrace{\frac{\deg(u)}{2|E|}}_{\text{pick } u} \times \underbrace{\frac{1}{\deg(u)}}_{\text{select } v} = \frac{1}{2|E|} = \text{prob of picking random edge}$$

procedure ① is essentially a 1-step random walk. by repeating this step, we go on longer and longer walks. as we repeat, the distribution of the end pt. of our walk becomes π.

formally, let $t \in \mathbb{N}$. pick $u \sim \pi$ (pick it randomly). Do a random walk starting @ u taking t steps. distribution of v, the end pt. of the walk is π.

π is also known as the stationary distribution on vertices. Also known as limiting / invariant distribution.

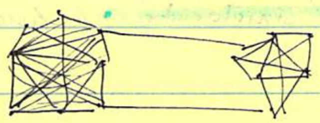
Suppose we don't start @ a random point. Say we start @ a pt. u_0. Under what cases will the dist. of v. not converge to π?

① G is disconnected

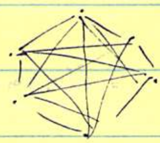
② G is bipartite (know which half you're in by even/odd steps)

When will it take a long time to converge to π? Consider

e.g.

1.  two cliques not "well connected"

vs.

2.  completely connected.

(1) should take longer than (2)

in e.g. (1), let $f = 1_S$ when S is one of the cliques.

$\mathcal{E}(f)$ is small.

roughly : fast convergence $\Rightarrow$ high $\mathcal{E}(f)$

slow convergence $\Rightarrow$ low $\mathcal{E}(f)$

Global Variance + Global Mean

let $u \sim \pi$, and $f: V \to \mathbb{R}$

(u randomly selected from π

$f(u)$ is a random variable w/ the following parameters.

mean $\quad E[f] := \underset{u\sim\pi}{E}[f]$

variance: $\quad \mathrm{Var}(f) := \underset{u\sim\pi}{E}\left[(f(u)-\mu)^2\right]$

$$= \underset{u\sim\pi}{E}\left[f^2(u)\right] - E[f]^2$$

$$= \frac{1}{2} \underset{u\sim\pi,\, v\sim\pi}{E}\left[\left(f(u)-f(v)\right)^2\right]$$

Known as the global variance b/c takes into account all u, v pairs, not just those w/ edges b/t them

Spectral graph theory compares local and global variances.

e.g. if for all f, $\mathcal{E}(f)$ larger $\mathrm{Var}(f)$, graph is an expander

(mix quickly)

Inner Products

Want a way to measure the "similarity" of functions on our graph.

let $f, g : V \to \mathbb{R}$

$\langle f, g \rangle_\pi = \underset{u \sim \pi}{E} [f(u)(g(u)]$ measures similarity.

similar to an inner product, but scaled by π.

e.g. $\frac{n}{n} \left\langle \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix}, \begin{bmatrix} g(v_1) \\ g(v_2) \\ \vdots \\ g(v_n) \end{bmatrix} \right\rangle = f(v_1)g(v_1) + f(v_2)g(v_2) + \ldots f(v_n)g(v_n)$

Note: $\langle f, g \rangle_\pi = \langle g, f \rangle_\pi$ (obvious) associativity

$\langle a \cdot f + g, h \rangle_\pi = a\langle f, h \rangle_\pi + \langle g, h \rangle_\pi$ linearity

$\langle f, f \rangle \geq 0$ & $= 0$ iff $f = 0$