Spectral Graph Theory II

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Last time, we proved that

$$L\phi = \lambda\phi \quad \text{for some } \lambda \in \mathbb{R}$$

where L is the normalized Laplacian operator. We call ϕ an eigenvector of L.

We found ϕ by maximizing $\mathcal{E}[f]$ under the constraint that $\|f\|_2^2 = 1$. To find another eigenvector of L, we need only add the constraint

$$f \perp \phi.$$

We can repeat this process, optimizing over a lower-dimensional space each time, until we have n eigenvectors

$$\phi_0, \phi_1, \ldots, \phi_{n-1}$$

with eigenvalues

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2.$$

The eigenvalues decrease with each iteration of the procedure above, since we are optimizing over a smaller set each time.

The last eigenvalue is zero because the last eigenvector is the 1-function:

$$L1 = I1 - K1 = 1 - 1 = 0.$$

The eigenvalues are uniquely determined, but not the eigenvectors. (For example, we could replace ϕ_k by $-\phi_k$.) We can fix a set of eigenvectors and then express any function f uniquely as

$$f = \hat{f}(0)\phi_0 + \hat{f}(1)\phi_1 + \cdots + \hat{f}(n-1)\phi_{n-1}.$$

The number of eigenvalues that are 0 are the number of connected components in G. Moreover,

$$Lf = \lambda_0 \hat{f}(0)\phi_0 + \lambda_1 \hat{f}(1)\phi_1 + \cdots + \lambda_{n-1}\hat{f}(n-1)\phi_{n-1}$$

It follows from orthonormality of the eigenvectors ϕ_i that

$$\langle f, g\rangle = \sum_{i=0}^{n} \hat{f}(i)\hat{g}(i).$$

And,

$$\mathcal{E}(f) = \langle f, Lf\rangle = \sum_{i>0} \lambda_i \hat{f}(i)^2.$$

$$\|f\|_2^2 = \sum_i \hat{f}(i)^2$$

$$E[f] = \langle f, 1\rangle.$$

$$\mathrm{Var}[f] = E[f^2] - E[f]^2$$
$$= \sum_{i>0} \hat{f}(i)^2.$$

Conductance and the 2nd Eigenvalue

Observe that

$$\min_{\mathrm{Var}[f]=1} \mathcal{E}[f] = \lambda_1.$$

Put another way,

$$\lambda_1 \mathrm{Var}[f] \leq \mathcal{E}[f].$$

This follows from the fact that $\mathrm{Var}[f]$ is the sum of non-negative numbers that add up to 1, whereas $\mathcal{E}[f]$ is the sum of those same numbers multiplied by non-negative eigenvalues λ_i. λ_1 is the smallest eigenvalue that is possibly non-zero.

Recall the definition of conductance:

Φ_G, the conductance of G, is

$$\min_{S:\, 0 < \mathrm{vol}(S) \leq \frac{1}{2}} \{\Phi(S)\}.$$

Note that a minimum bound on the conductance is

$$\Phi_G \geq \frac{1}{2} \min_{S: 0 < vol(S) \leq \frac{1}{2}} \left\{ \frac{\mathcal{E}[f]}{Var[1_S]} \right\},$$

because

$$Var[1_S] = vol(S)(1 - vol(S)) \geq \frac{1}{2} vol(S) = \frac{1}{2} \|1_S\|_2^2.$$

We can similarly express the second eigenvalue λ_1 as a minimization problem:

$$\lambda_1 = \min_{f: V \to \mathbb{R}, Var(f) \neq 0} \left\{ \frac{\mathcal{E}[f]}{Var[f]} \right\}.$$

Note: We can compute λ_1 efficiently, but computing the conductance of a graph is NP-hard. The values λ_1 and Φ_G are related to each other by Cheeger's Inequality.

Cheeger's Inequality

Theorem 1: $\frac{1}{2}\lambda_1 \leq \Phi_G \leq 2\sqrt{\lambda_1}$.

Proof: The first inequality follows from our formulation of λ_1 as a minimization problem:

$$\frac{1}{2}\lambda_1 = \frac{1}{2}\min_{f:V\to\mathbb{R},\, Var(f)\neq 0}\left\{\frac{\mathcal{E}[f]}{Var[f]}\right\}$$

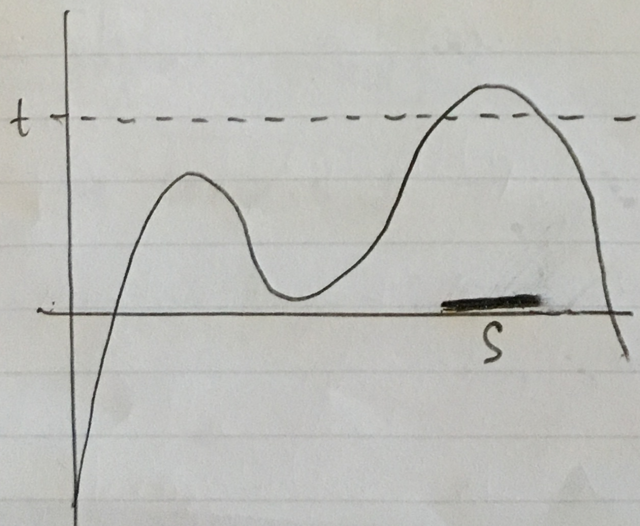
$$\leq \frac{1}{2}\min_{S:0<vol(S)\leq\frac{1}{2}}\left\{\frac{\mathcal{E}[1_S]}{\sqrt{Var[1_S]}}\right\}$$

$$\leq \Phi_G .$$

The second inequality is proved by the theorem below.

Theorem 2: Let f be a non-constant function such that $\mathcal{E}[f] \leq \lambda_1 Var[f]$. Then, $\exists S \subset V$ such that $0 < vol(S) \leq \frac{1}{2}$ and $\Phi(S) \leq 2\sqrt{\lambda_1}$.

In fact, there exists a set S of the form $S = \{u : f(u) > t\}$ or $S = \{u : f(u) < t\}$ for some value t. This result allows us to approximately solve the sparsest cut problem.



Definition: A function $g: V \to \mathbb{R}$ is convenient if $g \geq 0$ and $vol^\nu(\{u : g(u) \neq 0\}) \leq \frac{1}{2}$.

We can assume f is convenient in our proof of Theorem 2, but we lose a factor of 2 in the inequality $\mathcal{E}[f] \leq \lambda_1 \mathrm{Var}[f]$.

Lemma [Coarea Formula]: For a non-negative function $g: V \to \mathbb{R}^+$,

$$\int_0^\infty \Pr_{u \sim v}[(u,v) \text{ crosses } g^{>t} \text{ boundary}]\, dt = \mathop{E}_{u \sim v}[|g(u) - g(v)|]$$

where $g^{>t} = \{u : g(u) > t\}$.

Proof:

$$\int_0^\infty \Pr_{u \sim v}[(u,v) \text{ crosses } g^{>t} \text{ boundary}]\, dt$$

$$= \int_0^\infty \Pr[t \text{ is between } g(u) \text{ and } g(v)]\, dt$$

$$= \mathop{E}_{u \sim v}\left[\int_0^\infty 1\{t \text{ is between } g(u) \text{ and } g(v)\}\, dt\right]$$

$$= \mathop{E}_{u \sim v}[|g(u) - g(v)|].$$

Corollary: Let g be convenient. Then

$$E[|g(u) - g(v)|] \geq 2 \Phi_G \mathop{E}_{u \sim v}[g(u)].$$

Proof:

$$\mathbb{E}[|g(u)-g(v)|] = \int_0^\infty 2 \Pr_{u\sim v}[u\in S,\ v\notin S]\,dt$$
$$= \int_0^\infty 2\,\mathcal{E}[1_{g>t}]\,dt$$
$$= 2\int_0^\infty \Phi(g^{>t})\,\mathrm{vol}(g^{>t})\,dt$$
$$\geq 2\Phi_G \int_0^\infty \Pr_{u\sim\pi}[u\in g^{>t}]\,dt$$
$$= 2\Phi_G \int_0^\infty \Pr_{u\sim\pi}[g(u)>t]\,dt$$
$$= 2\Phi_G\, E_{u\sim v}[g(u)]$$

Proof of Theorem 2:

Assuming g is convenient and not always zero, by the 'corollary' above,

$$\Phi_G \leq \frac{E_{u\sim v}[|g(u)-g(v)|]}{2E_{u\sim v}[g(u)]}.$$

Let f be a convenient function such that

$$\frac{\mathcal{E}[f]}{\mathrm{Var}[f]} \leq 2\lambda_1.$$

(Recall that we lose a factor of 2 here by requiring that f be convenient.)

Next, let $g = f^2$. g is still non-negative and $g(u) = 0$ iff $f(u) = 0$, so g is also convenient.

$$\mathop{\mathbb{E}}_{u \sim v}\left[\,|f(u)^2 - f(v)^2|\,\right] = \mathop{\mathbb{E}}_{u \sim v}\left[\,|f(u) - f(v)| \cdot |f(u) + f(v)|\,\right]$$

$$\leq \sqrt{\mathop{\mathbb{E}}_{u \sim v}\left[(f(u) - f(v))^2\right]} \sqrt{\mathop{\mathbb{E}}_{u \sim v}\left[(f(u) + f(v))^2\right]}$$

$$\leq \sqrt{2\mathcal{E}[f]} \sqrt{\mathop{\mathbb{E}}_{u \sim v}\left[2f(u)^2 + 2f(v)^2\right]}$$

$$\leq \sqrt{4\lambda_1 \mathop{\mathbb{E}}_{u \sim v}\left[f(u)^2\right]}\; 2\sqrt{\mathop{\mathbb{E}}_{u \sim v}\left[f(u)^2\right]}$$

$$= 4\sqrt{\lambda_1} \mathop{\mathbb{E}}_{u \sim v}\left[f(u)^2\right].$$

Thus,

$$\Phi_G \leq \frac{\mathop{\mathbb{E}}_{u \sim v}\left[\,|f(u)^2 - f(v)^2|\,\right]}{2\mathop{\mathbb{E}}_{u \sim v}\left[f(u)^2\right]}$$

$$\leq \frac{4\sqrt{\lambda_1} \mathop{\mathbb{E}}_{u \sim v}\left[f(u)^2\right]}{2\mathop{\mathbb{E}}_{u \sim v}\left[f(u)^2\right]}$$

$$= 2\sqrt{\lambda_1}.$$

Now, using Theorem 1, how can we efficiently compute an approximately minimal conductance set $S \subseteq V$?

Given G, it is possible to compute a value $\tilde{\lambda}_1$ and a function $f: V \to \mathbb{R}$ such that

a) $|\tilde{\lambda}_1 - \lambda_1| \leq \epsilon$,

b) $\mathcal{E}[f] \leq \tilde{\lambda}_1 \operatorname{Var}[f]$

In time either

$$\tilde{O}(n) \cdot \frac{1}{\epsilon}$$

or

$$\operatorname{poly}(n) \cdot \log\left(\frac{1}{\epsilon}\right).$$