Hardness of Approximation II

Simon Zheng

1. Unique games conjecture
2. Review: Goemans-Williamson for MAX-CUT
3. UG Hardness of MAX-CUT$_{\frac{\arccos \rho}{\pi}+\varepsilon,\ \frac{1-\rho}{2}-\varepsilon}$
   - reduction
   - completeness

last lecture: decision problem for MAX-E3LIN

$\leq \frac{1}{2}+\varepsilon$ vs $\geq 1-\delta$ is NP-Hard

Notation: MAX-E3LIN$_{\frac{1}{2}+\varepsilon,\ 1-\delta}$

$$\text{MAX-E3LIN}_{\frac{1}{2}+\varepsilon,\ 1-\delta} \geq_p \text{LABEL-COVER}_{\eta,1}$$

goal of today show

$$\text{MAX-CUT}_{\frac{\arccos \rho}{\pi}+\varepsilon,\ \frac{1-\rho}{2}-\varepsilon} \geq_p \text{ULC}(m)_{\delta,1-\delta}$$

↑ unique label cover

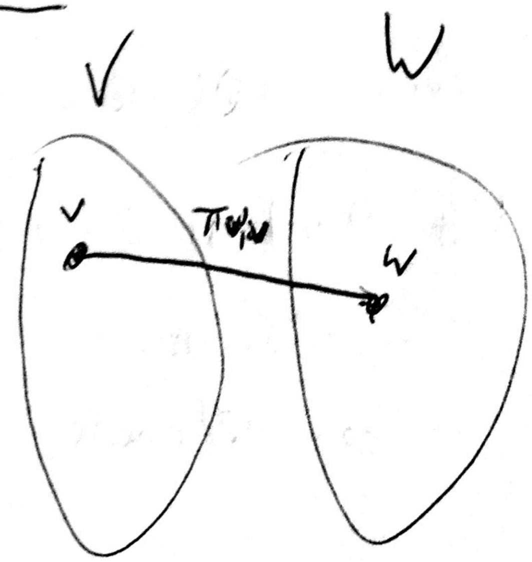
$\Rightarrow$ goemans-williamson algorithm is optimal for MAX-CUT

① Unique games Conjecture

<u>def</u> <u>unique label cover ULC(m)</u>

<u>input</u> - bipartite graph $(V \cup W, E)$

- set of labels Σ, where $|\Sigma| = m$
- edge functions: for all $(v,w) \in E$, there exists bijection $\pi_{w,v} : \Sigma \to \Sigma$

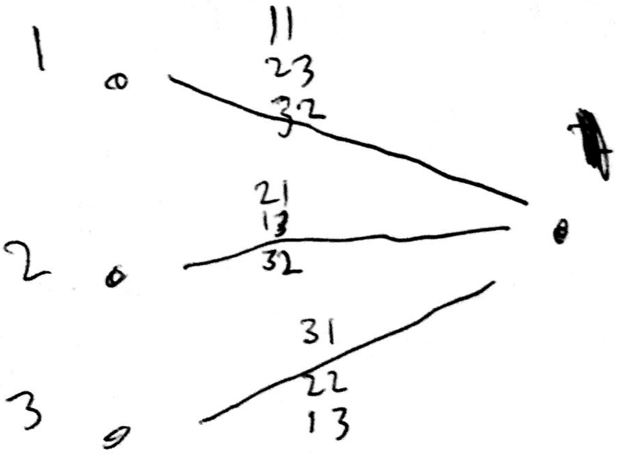


↑ makes this problem the unique label cover as opposed to label cover, in which $\pi_{v,w}$ not necessarily bijection

<u>output</u> label all vertices $\sigma : V \cup W \to \Sigma$ such that the fraction of satisfied edges is maximized

satisfy edge $(v,w) \iff \pi_{w,v}(\sigma(w)) = \sigma(v)$

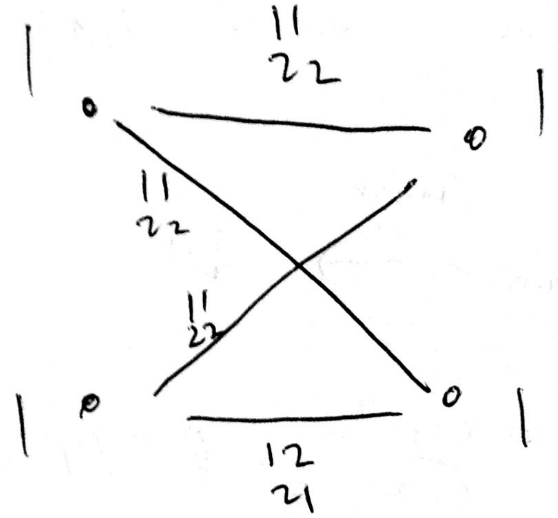
<u>ex</u> $\Sigma = \{1, 2, 3\}$



all edges satisfied

ex

$\Sigma = \{1,2\}$



can satisfy at most $\frac{3}{4}$ edges

remark $ULC_{\delta,1}$ is in P.

insight: since $\pi_{v,w}$ are bijections, given the label of a vertex, the labels of its neighbors are fixed. For each connected component of the bipartite graph, we can try all possible labels for a starting vertex and then search the rest of the component. This search is $\text{poly}(m, |V \cup W|)$.

unique games conjecture (Khot 2002) (UGC)

For all $\delta > 0$, there exists m such that $ULC(m)_{\delta, 1-\delta}$ is NP-hard.

Note all we did is change 1 to $1-\delta$ from the remark.

if conjecture is true, then a wide range of known approximations for optimization problems such as MAX-CUT, MAX-2SAT, VERTEX-COVER are all optimal!

we will show that Goemans-Williamson is optimal for MAX-CUT, assuming UGC.

(2) Review: Goemans-Williamson for MAX-CUT

MAX-CUT ~~integer~~ quadratic linear program $\xrightarrow{\text{relax}}$ SDP

let $\alpha_{GW} = \frac{2}{\pi} \min_{-1 < \rho < 1} \frac{\arccos \rho}{1-\rho} \approx 0.879$

occurs at $\rho \approx -0.689$

then

Goemans-Williamson is α_{GW}-approx. for MAX-CUT.

(3) UG Hardness of MAX-CUT

<u>theorem</u> let $\varepsilon > 0$, $-1 < \rho < 0$. ~~There exist~~ There exist δ and m such that

$$\text{MAX-CUT}_{\frac{\arccos \rho}{\pi} + \varepsilon,\ \frac{1-\rho}{2} - \varepsilon} \geq_p \text{ULC}(m)_{\delta, 1-\delta}$$

<u>Corollary</u> Goemans-Williamson is optimal, assuming UGC.

<u>proof</u> NP Hard to approximate MAX-CUT factor

$$\geq \frac{\frac{\arccos \rho}{\pi} + \varepsilon}{\frac{1-\rho}{2} - \varepsilon} > \alpha_{GW}$$

proof of thm 3 parts : reduction, completeness,

soundness

↓

next lecture

this lecture

pick δ sufficiently small (will specify how small later) and then m such that $ULC(m)_{\delta, 1-\delta}$ is NP-Hard (assuming UGC).

we have black box that solves $\text{MAX-CUT}_{\frac{\arccos \rho}{\pi}+\varepsilon,\ \frac{1-\rho}{2}-\varepsilon}$

and we have instance of $ULC(m)_{\delta, 1-\delta}$.

let $G = (V \cup W, E)$ be bipartite graph in $ULC(m)$.

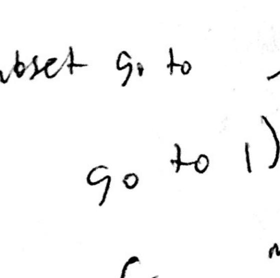
~~Notation:~~

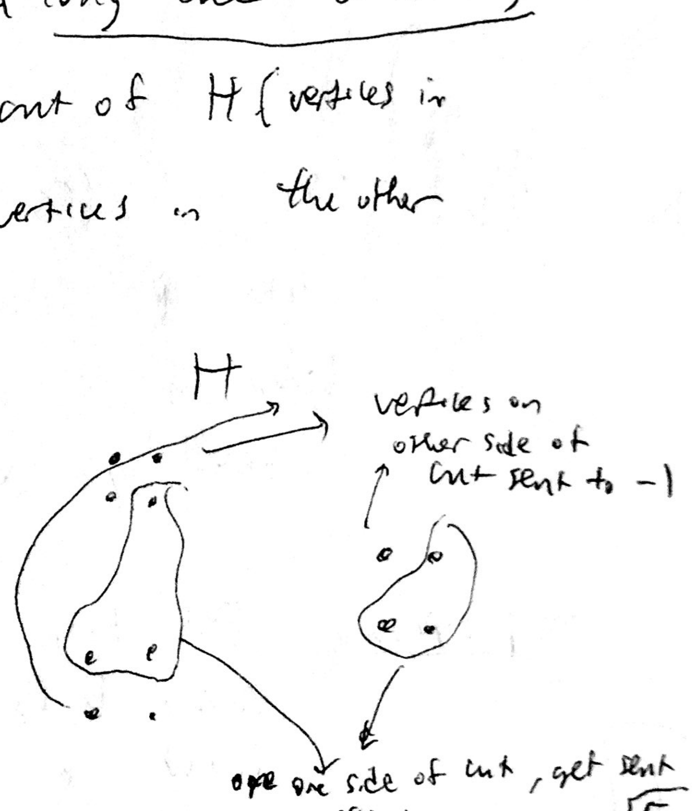
create new graph H with vertex set $(V \cup W) \times \{-1,1\}^m$.

Associate to each $u \in V \cup W$ a <u>long code encoding</u>

$f_u : \{-1,1\}^m \to \{-1,1\}$ given a cut of H (vertices in one subset go to -1, and vertices in the other subset go to 1).

ex G $m = 2$





~~Note fraction of CUT on graph where ea. vertex is sent to +1 or −1 (representing cut) denoted f(x) or f(x) = −1 for x ∈ V is equal to~~

First try at reduction

Notation:

$x, y \in \{-1,1\}^m$, $\pi : [m] \to [m]$ bijection

- xy coordinate-wise multiplication: $(xy)_i = x_i y_i$
- $x \circ \pi = (x_{\pi(1)}, \dots, x_{\pi(m)})$

(1) randomly pick $(v, w) \in E$ with permutation π

(2) pick $x \in \{-1,1\}^m$ uniformly

(3) pick $\mu \in \{-1,1\}^m$ ~~ea. coordinate~~ independently pick ea. coord.

$$\mu_i = \begin{cases} -1 & \text{w.p. } \frac{1-\rho}{2} \\ 1 & \text{w.p. } \frac{1+\rho}{2} \end{cases}$$

(4) test $f_v(x) \neq f_w((x\mu) \circ \pi)$

if true, put edge between vertex in v's hypercube labeled by x and vertex in w's hypercube labeled by $(x\mu)\circ\pi$.

Problem: resulting graph bipartite: could make all $f_v \equiv 1$ for $v \in V$ and all $f_w \equiv -1$ for $w \in W$ and test always passes!

actual reduction Only look at ~~one part of~~

long codes of ~~one~~ vertices from one of V, W

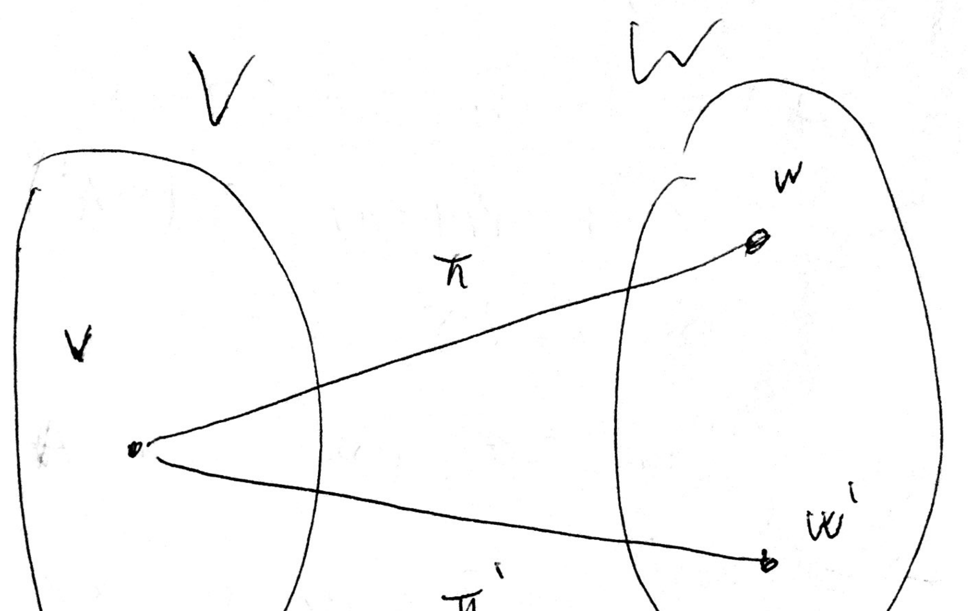
1. pick $v \in V$ uniformly, and two edges (v,w), (v,w') uniformly at random with permutations π, π' respectively

2. pick $x \in \{-1,1\}^n$ uniformly

3. pick $\mu \in \{-1,1\}^m$ where ea. coordinate i independently chosen as

$$\mu_i = \begin{cases} -1 & \text{w.p. } \frac{1-\rho}{2} \\ 1 & \text{w.p. } \frac{1+\rho}{2} \end{cases}$$

4. test $f_w(x \circ \pi) \neq f_{w'}((x\mu) \circ \pi')$



$(v,w), (v,w')$ satisfied $\iff$ $\pi'(\sigma(w')) = \sigma(v)$, $\pi(\sigma(w)) = \sigma(v)$.

Completeness

Suppose ULC(m) has assignment $\sigma : V \cup W \to \Sigma$ satisfying $\geq 1-\delta$ fraction of edges. Let each f_u be proper long encoding of $\sigma(u)$: the dictator function. $f_u(x) = x_{\sigma(u)}$ for all $u \in V \cup W$, $x \in \{-1,1\}^m$.

~~Pf~~

Edges (v,w), (v,w') from test

~~Pr (both edges satisfied)~~

A = "(v,w) satisfied"

A' = (v,w') satisfied

By union bound,

$$\Pr(A \cap A') = 1 - \Pr(\neg A \cup \neg A')$$
$$\geq 1 - \Pr(\neg A) - \Pr(\neg A')$$
$$\geq 1 - 2\delta$$

Now look at prob. test passes given $A \cap A'$.

$$f_w(x \circ \pi) = (x \circ \pi)_{\sigma(w)} = x_{\pi(\sigma(w))} = x_{\sigma(v)}$$

$$f_{w'}((x\mu) \circ \pi') = ((x\mu) \circ \pi')_{\sigma(w')} = (x\mu)_{\pi'(\sigma(w'))}$$
$$= (x\mu)_{\sigma(v)} = x_{\sigma(v)} \mu_{\sigma(v)}$$

$$\Pr\left[f_w(x\circ\pi) \neq f_{w'}((x\mu)\circ\pi')\right]$$

$$= \Pr\left[\cancel{x}\,\sigma(v) \neq x_{\sigma(v)}\,\mu_{\sigma(v)}\right]$$

$$= \Pr\left[\mu_{\sigma(v)} = -1\right]$$

$$= \frac{1-\rho}{2}$$

Thus

$$\Pr(\text{test passes}) \geq \left(\frac{1-\rho}{2}\right)(1-2\delta)$$

$$\geq \left(\frac{1-\rho}{2}\right) - \varepsilon$$

if $\delta \leq \frac{\varepsilon}{2}$. $\blacksquare$

shows completeness, next lecture will show soundness.